COLLEGE PHYSICS

FOR STUDENTS IN HIGHER ACADEMIES AND COLLEGES

BY

JOHN OREN REED, PH.D

PROFESSOR OF PHYSICS IN THE UNIVERSITY OF MICHIGAN

PART I

MECHANICS, ACOUSTICS AND OPTICS

ANN ARBOR

GEORGE WAHR, PUBLISHER

1907

The Ann Arbor Press, Printers

PREFACE

In the preparation of this book the needs of two distinct classes of students have been kept constantly in mind. First those who may pursue the study of Physics as a means to general culture, and as a foundation for further scientific study, and second those who may desire a knowledge of general Physics in view of its immediate application in the study of Medicine, Chemistry or Dentistry.

It has been the intention so to present the subject as to meet the needs of both classes, so far as it can be done in a course of lectures upon General Physics supplemented by regular recitations upon the text. Special stress has been laid upon the solution of problems in Mechanics and Optics, and to this end a considerable number of representative physical problems have been added.

In the preparation of a text for the use of first year students in the University, no effort has been made to secure either completeness or originality of treatment. The author has drawn freely upon standard works upon Physics, notably upon those of Hastings and Beach and Carhart, in English, and of Warburg, Müller-Pouillet, and Drude, in German. The aim has been so to connect logical development with practical application in each branch of the subject, as to awaken and stimulate a desire for further study of this fundamental and far-reaching science. If this little volume shall have in any degree contributed to this end, it will have repaid many fold the labor expended in its preparation.

J. O. R.

PREFACE TO SECOND EDITION.

In the preparation of the second edition, the chapters on mechanics have been materially amplified, and in some cases entirely rewritten. In the part on Acoustics the topics have been rearranged and a number of problems added. In its new form it is hoped the book will continue to merit the favor unexpectedly shown it in the old.

My special thanks are due to my friend Professor W. J. Hussey, for assistance in reading the proof, and for many valuable suggestions during the progress of the work. For the errors which remain, I alone am responsible.

Ann Arbor, Michigan. J. O. R.

January, 1907.

CONTENTS

MECHANICS

CHAPTER I

INTRODUCTORY

CHAPTER II

MOTION AND FORCE

CHAPTER III

TYPES OF MOTION

CHAPTER IV

WORK AND ENERGY

CHAPTER V

MECHANICS OF A RIGID BODY

CHAPTER VI

ELASTICITY

CHAPTER VII

MECHANICS OF FLUIDS

(A) FLUIDS AT REST

(B) FLUIDS IN MOTION

CHAPTER VIII

MOLECULAR MECHANICS

ACOUSTICS

CHAPTER XI.

NATURE OF SOUND

Chapter X

VELOCITY OF SOUND

Chapter XI

REFLECTION AND SUPERPOSITION OF SOUND WAVES

Chapter XII

MUSICAL RELATIONS

Chapter XIII

VIBRATORY PHENOMENA AND RESONANCE

CHAPTER XIV

VIBRATION OF AIR IN PIPES AND CAVITIES

OPTICS

CHAPTER XV

FUNDAMENTAL PHENOMENA

GEOMETRICAL OPTICS

CHAPTER XVI

PHENOMENA OF REFLECTION

CHAPTER XVII

PHENOMENA OF REFRACTION

CHAPTER XVIII

PRISMS AND LENSES

CHAPTER XIX

DISPERSION

CHAPTER XX

OPTICAL INSTRUMENTS

PHYSICAL OPTICS

CHAPTER XXI

VELOCITY OF LIGHT

CHAPTER XXII

INTERFERENCE

CHAPTER XXIII

DIFFRACTION

CHAPTER XXIV

RESOLVING POWER OF OPTICAL INSTRUMENTS

CHAPTER XXV

POLARIZATION

CHAPTER XXVI

EXPERIMENTAL DEMONSTRATIONS

COLLEGE PHYSICS

MECHANICS

CHAPTER I

INTRODUCTORY

1. Science and Natural Law. Physical Science is concerned with the deduction of the laws of phenomena in the inanimate world.

The natural tendency of the human mind is to try to arrange the facts of daily observation according to some rational plan; to subject them to some general rule; in short to explain them. A new fact is considered as explained when it is shown to be in accord with previous knowledge. Thus daily experience shows that all bodies, such as wood, stone, lead, water, etc., if unsupported fall to the ground, or if supported, they exert pressure upon the support; in other words they have weight. Torricelli, recognized as the cause of the pressure of the air, the fact already known to Galileo, that even gases have weight, and showed that the ocean of air presses upon the earth's surface because of its weight. In this way the phenomenon of atmospheric pressure was brought into harmony with the facts of previous knowledge, and with the general proposition that all bodies have weight.

Such a proposition is called a law of nature. It has been found to be true in all cases observed, and while it may be included in some more general proposition, it can never be shown to be false. Science has been defined as "a body of generalizations so irrefragably true, that while they may be subsequently included in some larger generalization they can never be overthrown."

The term Physical Science embraces the related branches, Physics and Chemistry, whose boundaries are separated by no sharp line of demarcation, but overlap in many places and many laws are common to both sciences. Their ultimate problem is the constitution of matter, and their methods of attack are daily becoming more and more similar, the common basis being the study of matter in its relation to energy.

2. Matter. Matter may be defined as that which occupies space. A mass is a definite quantity of matter. Chemistry is occupied with the investigation of changes in the composition of matter. Its fundamental proposition is that of the "Conservation of matter." In accordance with this principle it is asserted that the quantity of matter in the universe is constant, and that by no human agency can matter be created or destroyed.

Physics is concerned with matter only in so far as it serves as a *carrier of energy*. The fundamental proposition of Physics is the "Conservation of energy." This proposition asserts that the quantity of energy in the universe is constant; that energy like matter is indestructible, and that although it may be transformed and transferred in an endless round of changes no energy is ever lost; the amount of energy remains the same. This does not mean that all the energy is available, or that it will remain so. Much of the energy at our disposal is wasted, in that it escapes in the form of uniformly diffused heat and is thereby rendered unavailable. It has not, however, been destroyed.

3. Inertia. Of the various properties of matter, such as extension, impenetrability, divisibility, porosity, compressibility, elasticity, weight and inertia, perhaps the most characteristic is that of inertia. Inertia is the resistance which matter offers to any change in its condition of rest or motion. Matter is powerless of itself either to move or to stop moving if once set in motion; moreover it resists any attempt to move it if at rest, or to stop it if in motion.

Illustrations of inertia are seen in the hammering of the water in a water pipe on suddenly closing the faucet; in the action of the hydraulic ram; or of the fly wheel of an engine. Familiar examples are also found in the stamping of snow from the feet, in the beating of dust from a carpet, in the motion of a

bicycle rider when his wheel strikes a stone, or in the case of a person who steps from a rapidly moving car while facing to the rear. More remarkable illustrations of inertia are seen in the action of dynamite when exploded upon the surface of a rock—the inertia of the air being sufficient to cause the rock to be pulverized by the sudden pressure; in the method of supplying locomotives with water while running at full speed, and in milling machinery in which rapidly revolving steel bars beat the grain to powder.

4. Fundamental Units and Measurements. The measurement of any concrete or physical quantity consists in comparing it with some quantity of the same kind assumed as the standard or unit. Its magnitude or measure is then stated in terms of that unit, and consists of two parts; a *numerical* part, and the part which *names the unit* with which it has been compared. Both these parts are needed to give an idea of the magnitude of the quantity. Thus we may give the length of a table as 4.57 *meters,* or as 4.57 *feet,* but no idea of the length of the table is possible until the *unit of length* is stated.

The fundamental concepts of Physics are those of space, mass and time, and most physical quantities may be expressed in terms of these. For this reason the units of *length, mass* and *time* are called *fundamental units,* and all other units expressible in terms of these are called *derived units.* Such a system of units is called an absolute system. In the system in common use among scientific men the unit of length is the *centimeter,* the unit of mass is the *gram,* and the unit of time is the *second.* This is usually called the C. G. S. system.

The *centimeter* is the one hundredth part of the standard meter. The standard meter is represented by the distance at 0° Centigrade, between two marks on a certain bar of metal kept in the Archives of Paris. The meter was originally intended to be the one ten-millionth part of an earth-quadrant from equator to pole. Subsequent measurements have shown this distance to be 10,000,880 meters. The term meter, therefore, refers to the bar of metal and has no relation to the shape or size of the earth. The relation between the centimeter and the inch is

$$1 \text{ in.} = 2.54 \text{ cms.}$$

The *gram* is the one thousandth part of a mass of metal called a kilogram, kept in the Archives of Paris. It was intended that the gram should represent the mass of one cubic centimeter of distilled water at the temperature of maximum density [4° C.]; although more exact determinations have shown this relation to be slightly in error, yet for all practical purposes we may regard the mass of one cubic centimeter of distilled water at 4° C. as equal to one gram. The relation between the kilogram and the pound is

$$1 \text{ kilo} = 2.2 \text{ lbs.}$$

The *second* is the unit of time employed in scientific measurement and may be defined as the $\frac{1}{86400}$ part of a mean solar day, where a mean solar day denotes the average time between the successive passages of the sun across the meridian, taken throughout the year.

It remains to be noted that while the units of mass, length and time are called fundamental units, there are employed other units not directly reducible to these though connected with them by certain constants which have been determined by experiment. Such are the units of temperature, of heat, and of luminous intensity.

5. Dimensional Formulae and Derived Units. It is frequently of advantage to express physical quantities in general terms, in order to show more clearly their relation to each other. In such cases we write the symbols $[M]$, $[L]$, and $[T]$, with the proper exponents, to show how the fundamental quantities of mass, length and time enter into the derived units in question. Such a formula is called a *dimensional* formula. Since the dimensional formula is concerned with the *nature* of the quantity, rather than with its magnitude, numerical coefficients do not appear. A few examples of derived units will illustrate.

Area. The area of any plane figure is proportional to the product of two of its linear dimensions; hence the dimensional formula for *any area* is $[L^2]$; i. e., *the square of a length.*

Unit area is 1 cm^2.

Volume. Since the volume of any solid is proportional to the product of three of its linear dimensions, the general formula

for a volume becomes $[L^3]$; that is, *the cube of a length.* Unit volume is 1 cm.^{3} or 1 cc.

Density. The density of a body is by definition, *the mass per unit volume.* It is found by dividing the mass of a body by its volume; hence the formula for density is

$$\frac{M}{L^3} \text{ or } [M L^{-3}]$$

Velocity. Velocity is the *time rate of motion.* A moving body passes over a space L (cms.), in a certain time T (secs.); hence the mean or average velocity is found by dividing the space by the time, or the formula is $[LT^{-1}]$. Unit velocity is a velocity of one centimeter per second.

Acceleration. When the motion of a body is not uniform the velocity is no longer constant but varies every instant. If this variation be uniform, then the *average rate of change of velocity* is found by dividing the difference between the initial and the final velocities, by the time during which the change occurred. Acceleration is defined as the *time rate of change of velocity;* its equation is

$$a = \frac{v - v_0}{t} \qquad (1)$$

where v_0 and v denote the initial and final velocities of the body. The dimensional formula for acceleration is $[L T^{-2}]$. Unit acceleration is an acceleration of 1 cm. per sec. per sec.

Momentum. Momentum is the quantity of motion possessed by a body and is the product of its mass and its velocity. Momentum is symbolized by $m v$, and its dimensional formula is $[M L T^{-1}]$. Unit momentum is possessed by unit mass moving with unit velocity.

Force. A body has no power to change its motion of itself. Any change in the motion of a body, either in magnitude or direction, must be due to some action upon the body, which we term a force. Force may be defined as the *time rate of change of momentum.* Its equation is

$$F = \frac{mv - mv_0}{t} \qquad (2)$$

The dimensions of force are $(M L T^{-2})$. The unit of force is the *dyne*. It is that force which will give unit mass unit acceleration.

C. **Circular Measure of Angles.** About the vertex C of the angle θ (Fig. 1), describe a circle with radius r, and denote the subtending arc, AB, by s. Then in circular measure the angle θ is defined by the equation

$$\theta = \frac{s}{r} \tag{3}$$

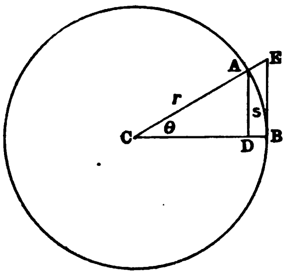


FIG. 1

If θ be taken as unity, then $s = r$. This unit angle is called *a radian*. A radian is that angle whose subtending arc is equal to the radius.

Again, from the point A drop a perpendicular to CB, and from B erect a perpendicular to CB to meet CA produced. Then in the two right triangles ACD, and ECB, we have

$$\sin \theta = \frac{DA}{CA} \tag{4}$$

and

$$\tan \theta = \frac{BE}{CB} \tag{5}$$

Noting the equality of CA and CB, we may write

$$\sin \theta = \frac{DA}{CB}, \; \theta = \frac{BA}{CB}, \; \tan \theta = \frac{BE}{CB} \tag{6}$$

or the angle θ lies between its sine and its tangent in value; that is,

$$\sin \theta < \theta < \tan \theta \tag{7}$$

As θ approaches 0, these values approach each other, and we may write, without serious error, for small angles

$$\sin \theta = \theta = \tan \theta \tag{8}$$

This approximation is frequently employed in physical formulae. The formulae are usually assumed as correct for values of θ under three degrees.

7. Trigonometrical Formulae. The following trigonometrical relations will find frequent application. In any right triangle ABC (Fig. 2), right angled at C, we have by definition

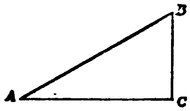


FIG. 2

$$\sin A = \frac{\textit{side opposite}}{\textit{hypothenuse}} = \frac{BC}{AB} \tag{9}$$

$$\cos A = \frac{\textit{side adjacent}}{\textit{hypothenuse}} = \frac{AC}{AB} \tag{10}$$

$$\tan A = \frac{\textit{side opposite}}{\textit{side adjacent}} = \frac{BC}{AC} = \frac{\sin A}{\cos A} \tag{11}$$

Also

$$\sin A = \cos (90^\circ - A) = \cos B \tag{12}$$

$$\sin^2 A + \cos^2 A = 1 \tag{13}$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \tag{14}$$

$$\sin 2A = 2 \sin A \cos A \tag{15}$$

The following values of the sine, cosine, and tangent of the various angles should be memorized.

θ	0°	30°	45°	60°	90°	180°
sine	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	0
cosine	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	-1
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0

8. Projection Upon Rectangular Axes. If we have a pair of rectangular axes XX', YY', (Fig. 3), then we can resolve any quantity capable of being represented by a right line, into two components parallel to the two given axes. Thus let OC represent such a line, making an angle θ with the axis of x. Then OA, the component of OC parallel to the axis of x is defined by the equation

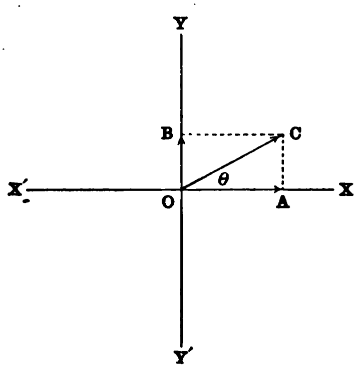


FIG. 3

$$OA = OC \cos \theta \tag{16}$$

Also

$$OB = OC \cos (90^\circ - \theta) = OC \sin \theta \tag{17}$$

These two components OA and OB, are called the *projections of OC* upon the axes of x and y respectively. The angle θ measures the difference in direction between the line OC and the axis, and is called the direction angle. *To project any line upon any other, multiply the line in question by the cosine of the direction angle.*

9. Curvature. The direction of motion of a body at any point in a curved path is the tangent to the curve at that point. Since the direction in which the body is moving constantly changes, it is of importance to know the rate at which the direction of motion changes. *Curvature is defined as the rate of change in direction per unit length of the curve.*

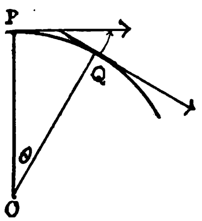


FIG. 4

In the case of a point describing the circular arc PQ, (Fig. 4), the change in direction is the angle included between the tangents at these points. This angle is in turn equal to the angle θ, included between the radii OP and OQ. But by equation (3)

$$\theta = \frac{s}{r}$$

hence the curvature is

$$\left(\frac{\theta}{s} = \frac{1}{r} \right) \tag{18}$$

or the *curvature* at any point on a curve is the *reciprocal* of the *radius* of the *circle* which most nearly coincides with the curve at that point. In the circle the curvature is constant; it is variable in the ellipse or the parabola, and zero in a straight line.

10. Vectors and Scalars. Certain physical quantities such as velocities, forces, accelerations, electromotive forces, electric currents, etc., are of such a nature that they require both a number and a direction to define them completely. Such quantities are called *directed quantities or vectors.* These quantities may be represented both in magnitude and direction by a straight line.

Other physical magnitudes such as mass, density, energy, power, electrical resistance, etc., have no relation to direction, and are completely described by a single numerical value. Such quantities are called *scalar quantities or scalars.*

11. Addition and Subtraction of Vectors. Given two vectors, p and q, represented in magnitude and direction by the straight lines AB and BC (Fig. 5), their sum r, may be determined and calculated as follows: From the point C drop a perpendicular upon AB produced,

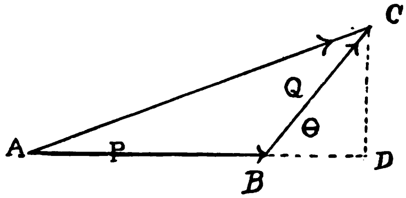


FIG. 5

and let the angle θ be the difference in direction between the two vectors. Then AC is the sum of the vectors p and q; for

$$\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2 \tag{19}$$

or

$$r^2 = [p + q \cos \theta]^2 + q^2 \sin^2 \theta \tag{20}$$

whence

$$r^2 = p^2 + q^2 + 2\,pq \cos \theta \tag{21}$$

It is to be noted that this formula covers all possible cases arising from different values of θ. Thus for

$\theta = 0°$, $r = p + q$; for $\theta = 90°$, $r^2 = p^2 + q^2$; $\theta = 180°$, $r = p - q$

The solution may be effected graphically by laying off and connecting in order the lines AB and BC, representing the vectors p and q; then the line AC represents the vector sum of p and q, where the points A, B, and C may be any points whatever.

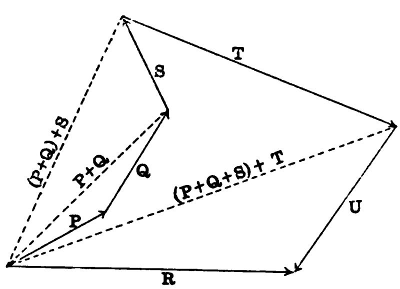


FIG. 6

12. Summation of Any Number of Vectors. The foregoing graphical method may be extended to the case involving any number of vectors. Thus the resultant R, (Fig. 6), of the vectors P, Q, S, T and U, is represented both in magnitude and direction by the line drawn from the beginning of the first to the end of the last, the vectors being added in any order whatever.

The numerical value of the resultant of any number of vectors is readily calculated by projecting the vectors upon the axes of x and y, adding the various components upon each axis, and then combining the x and y components. Thus if, (Fig. 7), p_1, p_2, ... p_n, etc., be the various vectors, making angles a_1, a_2, ... a_n, with the axis of x, then the sum of the x components is

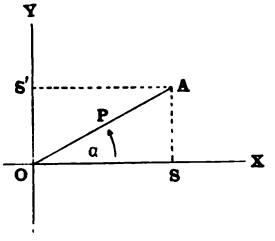


FIG. 7

$$x_1 + x_2 + \ldots\ldots x_n = p_1 \cos a_1 + p_2 \cos a_2 + \ldots\ldots p_n \cos a_n \qquad (22)$$

or

$$X = \Sigma\, p \cos a \qquad (23)$$

and similarly

$$Y = \Sigma\, p \sin a \qquad (24)$$

whence

$$R^2 = X^2 + Y^2 \qquad (25)$$

and the angle ϕ between the resultant and the axis of x is given by the equation

$$\tan \phi = \frac{Y}{X} \qquad (26)$$

CHAPTER II

MOTION AND FORCE

13. Motion. Motion is a change of position occurring in time. Displacement is a change of position without regard to time. All motion is purely relative; absolute rest is unknown in the universe. Motion embodies the two conceptions of space and time, and is to be studied under the two aspects of a change in the *magnitude* of the motion, and a change in the *direction* of the motion.

Motions may be divided (a) as regards direction, into rectilinear, curvilinear and vibratory motions; (b) as to kind, they may be divided into uniform, uniformly accelerated, and simple harmonic motions.

14. Force. Whenever a body has its condition of motion changed, either in magnitude or in direction, this change is attributed to some action upon the body which we term a *force.* A force is thus considered as an action upon a mass, and is measured by the product of the mass and the acceleration conferred upon the mass. Force is thus the product of Mass and Acceleration, or

$$F = M a \tag{27}$$

The C. G. S. unit of force is the *dyne.* A dyne is that force which will give a gram mass an acceleration of one centimeter per sec. per sec., or it is that force which will give to unit mass unit acceleration. The intensity of a force is the force exerted upon unit mass. Thus the intensity of the force of gravity is 980 dynes, since it imparts to a gram mass an acceleration of 980 cms. per sec. per sec.

When a force acts for but a short time, as in the case of a blow or a collision, the effect is called an *impulse* and is meas-

ured by the product of the force and the time during which the force acts, or

$$\text{Impulse} = F\,t \tag{28}$$

The element of time is essential to the consideration of the effect of any force, as nothing short of an infinite force could produce an effect in zero time.

Again every force is to be considered as resulting from the mutual action of two bodies. Under this aspect the resultant action is called a *stress*. Thus a mass of 1 kilogram is attracted towards the earth and in turn attracts the earth with a force of 980,000 dynes. This mutual action tends to produce motion in the case of each body. The force by which a body is attracted toward the earth is called its *weight*, and its weight is the product of its mass and the acceleration due to gravity, or

$$W = M\,g \tag{29}$$

Since the value of g increases slightly from the equator to the pole, it follows that the weight of a body is not constant at different points upon the earth. The mass of the body, however, is constant.

Weight is not to be confused with inertia, which signifies the resistance which a body offers to being set in motion *in any direction*. Weight is the force opposing motion of the body *away* from the earth's center. The inertia of a mass is a constant and characteristic property; its weight is a function of the acceleration due to gravity.

15. Newton's First Law of Motion. We have seen that a body is powerless to acquire motion of itself, and equally incapable of coming to rest of itself, if in motion. Change of motion therefore, is always due to the action of a force. The equation $F = M\,a$, shows that force does not appear except in connection with matter. There is always a mass involved. Again the factor a, indicates that force is exerted *only while the motion is changing*. A steam engine exerts force in pumping water from a well, in that it sets the water in motion; gunpowder exerts force upon the cannon ball during the time the ball is passing from the breech to the muzzle of the gun. The ball in turn exerts force only while its motion is changing, *i. e.*, while it is smashing the

target or piercing the armor of the ship. A ball flying through space and encountering no resistance exerts no force; its motion is unchanged. A locomotive pulling a train with uniform velocity along a level track, exerts force sufficient to overcome friction and no more. A body in motion moves until some force stops it. This is all summed up in Newton's first law of motion.

"Every body continues in its state of rest or uniform motion in a straight line, except in so far as it is compelled to change that state by force impressed upon it."

This law is symbolized in the equation $F = M a$. If a be zero there is no force; hence there must exist either *rest* or *motion*. Again, a denotes the rate of change of velocity *either* in *magnitude or direction*; hence if a be zero there must exist either *rest* or *uniform motion in a straight line.*

All apparent exceptions to the action of this law are in reality but proofs of its truth. A stone thrown into the air does not move with uniform velocity in a straight line for a single instant; yet the reason is found in the impressed force of gravitation, which compels it to change that state.

16. Newton's Second Law of Motion. The second law of motion is:

"Change of motion is proportional to the moving force impressed, and takes place in the direction in which the force acts."

Newton explains this law as follows: "If a force generate any motion a double force will generate a double motion, a triple force a triple motion, whether they be applied simultaneously and at once, or gradually and successively. This motion, if the body were already moving, is either added to the previous motion, if in the same direction, or subtracted from it, if directly opposed, or compounded with it if the two motions are inclined at an angle."

This law is also embodied in the equation $F = M a$, and gives us a means of measuring either force or mass as the case may be.

Thus we judge of the mass of a body by the force necessary to set it in motion. We kick a barrel lying on the ground to see whether it is empty or not. If full it is started with difficulty; if empty it moves very readily. Again, suppose we have small

cubical blocks of cork, aluminum and lead each mounted upon a little car so as to move readily on a smooth table. We attach to each car a small spring balance and tie the balance to a rod by which we pull the blocks quickly along the table. The result will be that while all the blocks have practically the same acceleration, the balances will indicate by their stretch the kinetic reactions, *i. e.,* the relative masses, of the various substances. In this case, the *acceleration* being the same for all, the force is directly proportional to the mass.

If on the other hand, the blocks with their cars be placed upon a smooth table and be struck equal blows, as from a spring hammer, then the accelerations produced in the various blocks will afford a measure of their relative masses. The cork block would move off rapidly, the aluminum more slowly and the lead would scarcely be moved at all. We should conclude that the lead contains the most matter, the aluminum next, and the cork least. In this case the *force* is kept constant, and the accelerations vary inversely as the masses.

17. Newton's Third Law of Motion. The third law of motion states that

"To every action there is an equal and contrary reaction, or the mutual actions of two bodies are equal and opposite." In explanation of this law Newton adds, "Whatever presses or draws another body is pressed or drawn to the same extent by that body. If one press a stone with the finger, the finger is pressed by the stone. If a horse pull on a stone by a rope, the horse is pulled equally towards stone; ......and to the extent that the forward motion of the one is aided the forward motion of the other is impeded."

This law expresses the two-sided nature of every force. Force is always the mutual action of two bodies, the action of the one being equaled by the reaction of the other. This amounts to saying that forces always occur in pairs. Such a pair of forces is called a *stress*; if the forces act toward each other it is a *pressure*, if away from each other it is a *tension*.

Again no force can be exerted unless there be some *resistance* to overcome. There can be no action unless there be something to act upon which will, in its turn, react. The athlete prefers to jump from a slab of stone; he can not "rise" from a pile of

straw or a heap of cushions. For the same reason it is tiresome to walk in melting snow or loose sand.

If motion ensue as the result of the action of two bodies, then the law expresses the equality of the resultant motions; that is, if a force confer upon the masses m and m', velocities of v and v', then the law states that their *momenta are equal,* or

$$m\,v = m'\,v' \qquad (30)$$

Illustrations of this law are manifold. The explosive force of the powder drives the ball from the cannon; the two are shot apart, moving with velocities inversely as their masses. The recoil or "kick" of a gun is the greater the more nearly the masses of gun and projectile are made equal to each other.

The screw of a ship drives the water *backward* with a velocity as many times *greater* than the forward velocity of the vessel, as the mass of water moved is *less* than that of the vessel. Boats have been propelled by machinery which pumped water in at the bow and expelled it in a small stream under high velocity at the stern. The motions of fish in the water, and of birds in the air, the ascent of sky-rockets, and the action of rotary lawn-sprinklers are all explained in accordance with Newton's third law of motion.

CHAPTER III

TYPES OF MOTION

18. Uniform Motion. The simplest type of motion is uniform motion in a straight line. In this case the velocity, or space traversed in unit time, *remains constant.* Equal spaces are described in equal times. The *acceleration* is therefore zero. We may, from these conditions, write down the equation of uniform motion from definition. We have

$$\frac{s}{t} = \bar{v} = \text{a constant} \tag{31}$$

whence

$$s = \bar{v}t \tag{32}$$

These conditions involve the permanence of any motion once set up, and the absence of any force to maintain it.

19. Uniformly Accelerated Motion. In the case of uniformly accelerated motion the body moves with a constantly and uniformly increasing or decreasing velocity. It passes over unequal spaces in equal intervals of time. The velocity is no longer constant, but the acceleration, or the rate at which the velocity changes, is constant. Hence we may write

$$\frac{v - v_0}{t} = a = \text{a constant} \tag{33}$$

The force producing uniformly accelerated motion is also a *constant force,* since

$$F = M a$$

From equation (33) we see that v, the velocity at any time t, is

$$v = v_0 + a\,t \tag{34}$$

or the final velocity v, is equal to the initial velocity v_0, plus the change in velocity acquired in time t.

The average velocity $\bar{v}$, during this interval of time is, of course,

$$\bar{v} = \frac{v + v_0}{2} \tag{35}$$

where $\bar{v}$ denotes the constant velocity at which the body would have described the same space in time t. The expression for the space described is readily found from equation (31); thus we have

$$s = \bar{v}t = \frac{v + v_0}{2} \cdot t \tag{36}$$

or, replacing v by its value from equation (34) we have

$$s = v_0t + \frac{at^2}{2} \tag{37}$$

Finally, combining equations (33) and (36) to eliminate t, we have

$$v^2 = v_0^2 + 2as \tag{38}$$

It is further to be noted that the acceleration may be either positive or negative; in the latter case the acceleration and the motion are oppositely directed, and the acceleration becomes a retardation. In their general form the equations of uniformly accelerated motion become

$$\begin{aligned} v &= v_0 \pm at \\ s &= v_0t \pm \frac{at^2}{2} \\ v^2 &= v_0^2 \pm 2as \end{aligned} \tag{39}$$

20. Freely Falling Bodies. In the case of a freely falling body equations (39) become, on substituting for the general acceleration a, the acceleration due to gravity ($g = 980$ cms. per sec. per sec.),

$$\begin{aligned} v &= v_0 \pm gt \\ s &= v_0t \pm \frac{gt^2}{2} \\ v^2 &= v_0^2 \pm 2gs \end{aligned} \tag{40}$$

In case the body start from rest, v_0 is zero, and all terms containing it disappear, hence

$$v = gt$$
$$s = \frac{gt^2}{2} \qquad (41)$$
$$v^2 = 2gs$$

It is to be noted that in case of a body starting from rest the velocity is proportional to the time, the space described is proportional to the square of the time, and the space described in the consecutive seconds varies as the odd numbers 1, 3, 5, 7, etc., as may be readily shown.

If a body be thrown vertically upward with an initial velocity v_0, the motion and the acceleration are oppositely directed and the lower sign is to be used in equations (40). The time of ascent and the height to which a body thrown vertically upward will rise, are found by setting the final velocity, $v = 0$, the value it assumes at the highest point. Then we have

$$t = \frac{v_0}{g} \qquad (42)$$

$$s = \frac{v_0^2}{2g} \qquad (43)$$

These values of t and s are the same as would be required to produce a velocity v_0, in the case of a body falling freely from rest.

21. Diminished Acceleration. Atwood's Machine. If we wish to study the laws of a falling body experimentally it is necessary to reduce materially the acceleration, as otherwise the motion is much too rapid to permit of accurate observation. This may be done in several ways. For example, the force of gravity acting upon a small mass may be applied to one or more large masses as well, in which case the resultant acceleration is correspondingly diminished. Thus suppose that a body of mass M be placed upon a perfectly smooth, horizontal table and have attached to it a light flexible cord passing over a frictionless pulley at the end of the table, and that from this cord there be suspended a small mass m. In this case the tension in the string will be that due to the weight of the mass m, and this force will produce

motion in the two masses M and m. If we denote the resulting acceleration by a, we may equate the two expressions for the force as follows:

$$m g = (M + m) a$$

whence

$$a = \frac{m}{M+m} \cdot g \tag{44}$$

In this way the value of a may be made what we will, and the motion rendered so slow as to allow us to study it at leisure.

In *Atwood's machine* the light flexible cord passes over a light wheel having a groove in its rim and mounted upon "friction wheels" so as to turn as freely as possible. Two equal masses M_1 and M_2, are hung to the ends of the cord and are in equilibrium in any position. If now there be placed upon one of the masses a small rider of mass m, then the resultant motion of the system is due to the force of gravity upon this small mass alone. If we set M equal to the combined masses M_1 and M_2, and neglect friction and the effect of the light wheel, we may compute the resulting acceleration a, at once from equation (44), and verify the result by actually observing the spaces passed over in one, two, three, or four seconds respectively. In case the rider is removed at any time, the acceleration becomes zero from that instant, and the motion becomes uniform motion. By means of special devices this may be accomplished and the machine may be used to verify all the conclusions represented in equations (41).

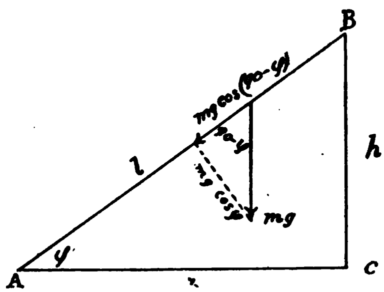


FIG. 8

22. Motion on an Inclined Plane. The inclined plane is another device for reducing the effect of gravity. Suppose a body slide without friction down a plane AB, (Fig. 8), making an angle ϕ with the horizon; it is required to find the equations of its motion. It is to be observed that the acceleration due to gravity effective in producing motion down the plane, is the component parallel to the surface of the plane. This component is readily found by pro-

jecting g upon the plane in accordance with Article 8. The effective component is seen to be $g \cos (90° - \phi)$ or $g \sin \phi$, and the equations of (40) become

$$v = v_0 \pm g \sin \phi \, t$$
$$s = v_0 t \pm \frac{g \sin \phi \, t^2}{2} \qquad (45)$$
$$v^2 = v_0^2 \pm 2 g \sin \phi \, s$$

Substituting for $\sin \phi$, its value h/l, and setting $v_0 = 0$, in the last equation of (45) we see that

$$v^2 = 2 g h$$

or the velocity acquired by a body starting from rest and sliding down the plane is the same as that which it would have acquired in falling through the vertical hight h.

23. Uniform Circular Motion. A body describing circular motion at a constant speed is yet under the action of a constant force, since the motion changes at every instant, not in *magnitude* but in *direction*.

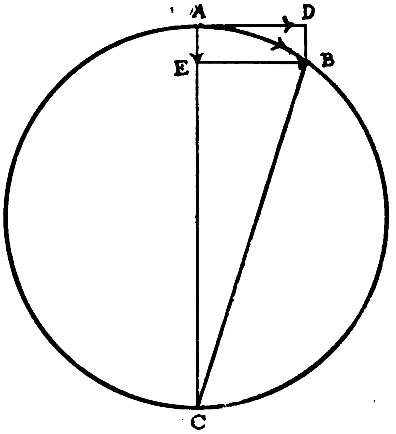


FIG. 9

In uniform circular motion therefore, there exists at every instant an acceleration toward the center. The value of this acceleration is readily calculated. Thus in Fig. 9, a particle performing uniform circular motion describes the arc AB in time t, with a velocity v. It has in the same time been deflected from a straight line AD, through the distance AE, or AE is the space described due to the constant force acting toward the center. If f be the acceleration toward the center then

$$A E = \frac{f t^2}{2} \qquad (46)$$

and

$$A B = v t \qquad (47)$$

since the motion is uniform.

If now the arc AB be taken very small, it will differ but little from a straight line, and the two triangles ABE and ABC are similar; hence

$$AB : AE :: AC : AB \tag{48}$$

or

$$\overline{AB}^2 = AE \cdot AC \tag{49}$$

whence

$$v^2t^2 = \frac{ft^2}{2} \cdot 2r \tag{50}$$

or

$$f = \frac{v^2}{r} \tag{51}$$

If T be the time of a complete revolution of the particle in the circle, then

$$v = \frac{2\pi r}{T} \tag{52}$$

and

$$f = \frac{4\pi^2 r}{T^2} \tag{53}$$

The quantity $\frac{2\pi}{T} = \omega$, is termed the angular velocity. Hence we may write

$$f = \omega^2 r \tag{54}$$

24. Applications of Uniform Circular Motion.

We have seen that in uniform circular motion a constant force is needed to keep the particle from flying off on a tangent. This stress is termed *Centripetal force,* and its measure is the product of the mass times the acceleration, or

$$F = m\frac{v^2}{r} \tag{55}$$

The tendency of the particle to fly off, or the reaction of the particle against being pulled out of a straight line, is called *Centrifugal force,* and its measure is also $m\,v^2/r$. These two so-called forces are in reality but the two aspects of a *stress,* the

one being the *action* of the moving mass, tending to move in a straight line and the other the *reaction* of the center.

Examples of applications of these forces are seen in the action of a boy's sling; in the tendency of a bicycle rider to fall when turning a corner sharply while riding at full speed; and in the necessity for raising the outer rail in railway curves. The shape of the earth is an example of this action on the plastic mass of the earth while in rotation. Other applications of this principle are seen in machines for drying clothes; for separating honey from the honey comb, or molasses from sugar. Cream separators, blood testers, lawn sprinklers, and the mechanical governors on steam engines all illustrate the same principle.

25. Simple Harmonic Motion. Simple harmonic motion is a motion in which the acceleration is proportional to the displacement of the moving body from its position of rest, and directed toward this position. It is an oscillatory motion, the body tracing the path over and over again. This is the most important type of motion to be studied, as it finds its applications in Sound, Light, and Electricity as well as in Mechanics.

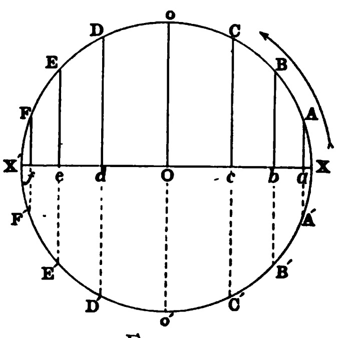


FIG. 10

Examples of simple harmonic motion are seen in the vibrations of the prongs of a tuning fork, of guitar strings or of a pendulum bob, if the amplitude be small; other examples of approximately this form of motion are found in the motion of the piston rod of an engine, of the shuttle of a sewing machine or of the sickle of a reaping machine.

Simple harmonic motion may also be defined as the apparent motion of a point describing uniform motion in a circle, when viewed at a great distance from the circle and in the plane of the circle. The apparent motion of the moons of Jupiter is a simple harmonic motion. These little bodies revolve about the planet in orbits nearly circular, and their motion as seen from the earth, is an oscillatory motion about the planet as a center.

For purposes of study, simple harmonic motion is usually treated in connection with the related case of uniform motion in a circle. Thus if we consider a point moving uniformly round a circle (Fig. 10), in the direction of the arrow, then the projections of the point upon any diameter of this circle will represent a simple harmonic motion, upon that diameter. The projection will describe a complete to and fro vibration upon the diameter, while the point on the circumference describes a complete revolution.

Thus while the point moves from *o′* through *C′*, *B′*, *A′*, to *X*, its projection falls successively upon *O, c, b, a,* and comes to rest for an instant at *X*. It then retraces its path to *O* as the point passes round to *o*. The point on the circle has described half a circumference and its projection has made half a vibration. As the point passes on through *D, E, F,* to *X′*, its projection swings through *d, e, f,* reaching the limit of its motion in *X′*, and returning completes the second half of its vibration as the point passes through *F′*, *E′*, *D′*, back to *o′*. The circle upon a diameter of which the simple harmonic vibration is supposed to occur is called *the circle of reference.*

26. Amplitude, Period, Phase, Epoch. The *amplitude* of the vibration is the radius of the circle of reference. The *period* of the vibration is the time required for the moving point to make a complete revolution in the circle or for its projection to execute a complete to and fro vibration upon the diameter.

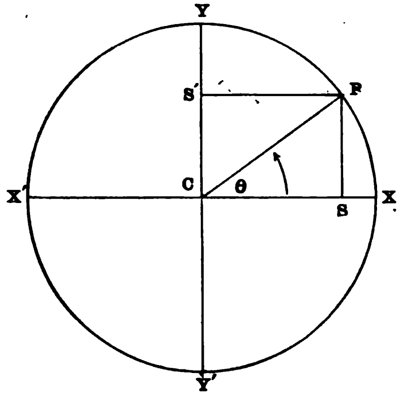


FIG. 11

Phase is that fraction of a period which has elapsed since the moving point last passed through the position of rest in the positive direction. Phase may be expressed either in time, or in an angle which varies as the time.

If ω represent the angular velocity of the radius vector *CP*,

then $\theta = \omega t$, is the angle swept out in time t; for, since the motion is uniform,

$$\frac{\theta}{2\pi} = \frac{t}{T}$$

or

$$\theta = \frac{2\pi t}{T} = \omega t \qquad (56)$$

The angle θ (Fig. 11), is called the *time angle, i. e.*, the angle swept out in time t, if time is counted from the instant the point P passes through X. In the case of motion on the axis of y, it is also the phase angle, since by definition, phase is measured from the instant at which the point P passes X, or its projection on the axis of y passes through C, in the positive direction.

For the same reason it is to be observed that in the case of motion on the axis of x, phase must be measured from the radius CY', since the point P is at Y' when its projection on the axis of x passes through C going in the positive direction. In this case the phase angle is $Y'CP$, or $\theta + 90°$. This shows that uniform circular motion is to be regarded as the resultant of two simple harmonic motions at right angles to each other, of the same period and amplitude, and differing in phase by 90°, or a quarter of a period.

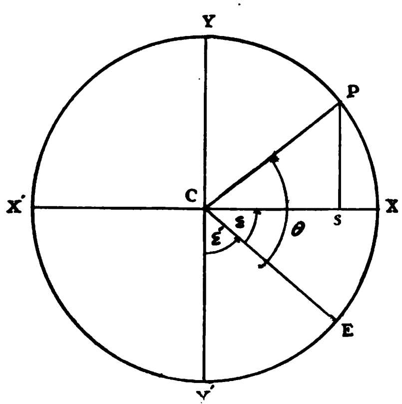


Fig. 12

Again if we should begin to count time from some other point, as E, (Fig. 12), then for the motion on the axis of y the time angle θ, is ECP, while the phase angle is XCP as before, or $(\theta - \epsilon)$. For the motion on the axis of x the time angle is ECP, but the phase angle is $Y'CP$ as before or $(\theta + \epsilon')$. The angles ϵ and ϵ' are called the **epoch angles.** The **epoch angle,** is the angular difference between the time angle and the phase angle, or it is the angle which must be added to or subtracted from the time angle to produce the phase angle.

It is further to be noted that while ϵ and T are constant for any specific case, the time angle θ, grows continuously from 0° to 2π with the growth of t from 0 to T.

27. Equations of Simple Harmonic Motion. In order to describe in mathematical terms the behavior of a point executing simple harmonic motion it is necessary to express its distance from the position of rest in terms of an angle that varies as the time. Thus in Fig. 13, the motion of the points s and s' is completely described if we write

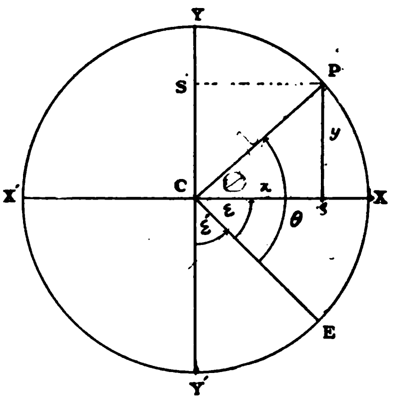


FIG. 13

$$x = r \sin Y'CP$$
$$y = r \sin XCP \qquad (57)$$

or

$$x = r \sin (\theta + \epsilon')$$
$$y = r \sin (\theta - \epsilon) \qquad (58)$$

As a special case we may assume the point E to coincide with X, then $\theta = XCP$, $\epsilon = 0$, and $\epsilon' = 90°$, and equations (58) become

$$x = r \cos \theta$$
$$y = r \sin \theta \qquad (59)$$

Again, since the acceleration in uniform circular motion is

$$f = \frac{4\pi^2 r}{T^2}$$

and *directed toward the center,* we have by projection upon the axes of x and y, the following expressions for the accelerations along these axes respectively:

$$f_x = -\frac{4\pi^2 r}{T^2} \cdot \cos \theta$$
$$f_y = -\frac{4\pi^2 r}{T^2} \cdot \sin \theta \qquad (60)$$

The negative sign denotes that the acceleration is always opposite in sign to the displacement. Equations (60) show that

the acceleration along the axis of x is a maximum when that along the axis of y is 0, and *vice versa*, since the one varies as the sine and the other as the cosine of the same angle.

If we substitute in equations (60) the values of sin θ and cos θ, we have

$$f_x = -\frac{4\pi^2 r}{T^2} \cdot \frac{x}{r} = -\frac{4\pi^2 \cdot x}{T^2}$$
$$f_y = -\frac{4\pi^2 r}{T^2} \cdot \frac{y}{r} = -\frac{4\pi^2 \cdot y}{T^2} \qquad (61)$$

which shows that the aceleration along the axis of x is proportional to x, and that along the axis of y is proportional to y, or that *the acceleration is proportional to the displacement*. This is the characteristic of simple harmonic motion.

28. Velocity of a Point Executing Simple Harmonic Motion. The velocity of a point executing simple harmonic motion may be readily calculated. Let P (Fig. 14), be the moving point, and let PV represent its velocity in the circle. Then V_x and V_y, the component velocities parallel to the axes of x and y, are found by projection as usual, and we have

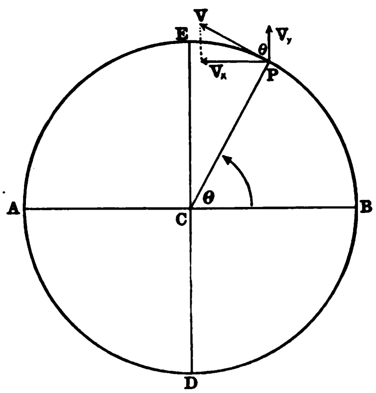


Fig. 14

$$V_x = -V \cos(90° - \theta)$$
$$= -V \sin \theta \qquad (62)$$
$$V_y = V \cos \theta$$

But V is the velocity in the circle, and is therefore $\frac{2\pi r}{T}$, hence

$$V_x = -\frac{2\pi r}{T} \cdot \sin \theta$$
$$V_y = \frac{2\pi r}{T} \cdot \cos \theta \qquad (63)$$

From equations (63) it appears that the velocity along the axis of x is a maximum when the velocity along the axis of y is zero, and *vice versa*. Also from comparison with (60) we see that when the velocity along the axis of x is a maximum the acceleration along x is zero; this means the velocity is greatest at the middle of the swing where the acceleration is zero, and

zero at the end of the swing where the acceleration is greatest.

It should also be noted that since the angular velocity ω, is a constant, the time of a vibration, T, is also a constant, and the vibrations of a body executing simple harmonic motion *are all performed in the same time, and are independent of the amplitude.* Such vibrations are said to be *isochronous.* This is seen in the constancy of the pitch of the musical note from a string or tuning fork, as the vibrations die away. This characteristic of simple harmonic motion is of high importance in the theory of the pendulum, in acoustics and optics.

29. The Curve of Sines. An important aid to the study of simple harmonic motion is found in the graphical method, whereby the moving body is made to trace its path upon some recording surface. In such cases the simple harmonic motion is compounded with a uniform motion in a straight line and the

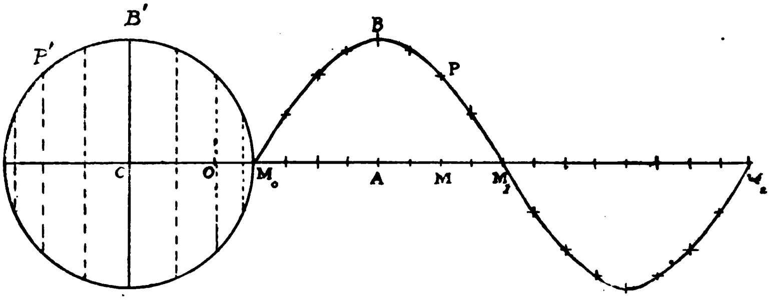


Fig. 15

resultant curve is called a *sine curve.* Such a curve is readily obtained as follows: Divide each quadrant of the circle, (Fig. 15), into four equal parts, each of which will thus correspond to the distance passed over by the point moving in the circle, in one sixteenth of a period. From the points of division drop perpendiculars to the x diameter; these perpendiculars will then denote the displacements of the point executing simple harmonic motion on the y axis, at the ends of the successive intervals of time, and are therefore proportional to the sines of the angles swept out in $T/16$, $2T/16$, $3T/16$, etc. Next lay off on the x diameter produced, a series of equal lengths to represent the spaces described in the same intervals, due to the uniform motion, and beginning

at M_0, erect perpendiculars equal to the y displacements at the corresponding times. Finally sketch a smooth curve through the extremities of these perpendiculars and the *sine curve* is the result.

Such a curve is readily obtained by allowing a tuning fork to trace, with a fine style, its vibrations upon a sheet of smoked glass or paper. The time interval between M_0 and M_1 corresponds to a half period, or the particles at M_0 and M_1 *differ in phase* by half a period, while the particles at M_0 and M_2 are in the *same phase.*

The distance M_0M_1, is a *half wave length,* and M_0M_2 is a *complete wave length,* where a *wave length* denotes the distance the motion has run forward in a single period T. Hence we have the important relation

$$\lambda = VT \qquad (64)$$

where λ denotes the wave length, V the velocity of the wave motion, and T the period of the simple harmonic vibration.

PROBLEMS

1. A body has a velocity of 60 miles an hour. Find its velocity in feet per second, and in cms. per second.

2. A body starts with a velocity of 640 cms. per second, and in ten minutes has a velocity of 3428 cms. per second. Find the acceleration.

3. A mass of 5000 gms. moves with a velocity of 4 meters per sec. Find its momentum.

4. A mass of 600 gms. starts from rest and in 6 secs. has a velocity of 36 meters per second. Find the force.

5. A vessel contains 400 cc. of sulphuric acid, density 1.8 gms. per cc. Find the mass of the acid.

6. Detroit is 38 miles from Ann Arbor. How long will it take to travel this distance at the rate of 5 kilometers per hour?

7. Express a mass of 65 lbs. in grams; a weight of 65 lbs. in dynes.

8. What force will in 10 minutes give a mass of 700,000 gms. a velocity of 124,000 cms. per sec.?

9. What is the force of gravity on a body whose mass is 700 lbs.?

10. Find the mass of 465 cc. of lead.

11. A wire guy rope, the tension of which is 12×10^8 dynes, makes an angle of 60° with the horizon. Find the vertical and the horizontal components of the tension.

12. Find the acceleration produced upon a mass of 4 grams by a force of 36 dynes.

13. A force of 60 dynes acts upon a body for one minute and imparts to it a velocity of 900 cms. per sec. Determine the mass of the body.

14. An engine draws a cage of mass 3000 kilos up a shaft at a uniform speed of 10 meters per second. Find the tension in the rope. What is the tension if the cage move with a uniform acceleration of 10 meters per sec. per sec.?

15. An elevator starts to descend with an acceleration of 300 cms. per sec. per sec.; find the pressure on its floor due to a man whose mass is 75 kilos. What would be his weight with respect to the elevator, if it started to *ascend* with the same acceleration?

16. A steamer whose velocity in still water is six miles an hour starts directly across a stream whose velocity is 10 miles per hour. Find the velocity of the steamer in crossing; also when it makes an angle of 30° with the current down stream.

17. A gun of mass 3000 kilos, placed upon a smooth horizontal plane, discharges a ball of 30 kilos mass at an elevation of 30° to the horizon; find the velocity of the gun's recoil in terms of velocity of the ball.

18. An inelastic mass of 900 kilos moving with a velocity of 30 meters per second, meets another, equal and similar mass moving in the opposite direction, at ten meters per second. Find velocity of total mass after impact.

19. A certain force acts upon m units of mass, and at the end of a second the mass is moving at the rate of 32 ft. per second. What velocity would be produced in 32 units of mass by the same force, in the same time?

20. How many dynes of force are required to give a mass

of 50 kilos a velocity of 12 meters per second, the force being supposed to act for exactly one second?

21. How many dynes are needed to give a gram a velocity of 9.81 meters per sec., if the force act for one second? What if it act for two seconds?

22. Compare the velocities produced in masses of 2 kilos, 750 gms. and one gram, by forces of 300,000, 112,500, and 150 dynes respectively.

23. Equal forces act upon the masses specified in the previous problem; find the accelerations produced.

24. A body falls freely from rest for 15.6 seconds. Find the final velocity and the distance traversed.

25. How long will it take a body to fall 650 ft., and what velocity will it acquire?

26. A body is thrown downwards with a velocity of 874 cms. per sec. Required its velocity and position at the end of 20 seconds.

27. A body is thrown vertically upwards with a velocity of 827 cms. per sec. How long will it continue to rise, and how high will it rise?

28. A body is thrown vertically upwards with a velocity of 697 cms. per sec. How long will it take to rise 195 cms., and what velocity will it then possess?

29. A ball is thrown vertically upward to a height of 150 ft. With what velocity did it leave the hand?

30. A mass of 876 gms. is attached to a spring balance which is carried upward at such a rate that the balance indicates 932 gms. What is the acceleration of the motion?

31. A mass of 162 kilos hanging by a perfectly flexible cord drags a mass of 973 kilos along the top of a smooth table. What is the acceleration of the system, and what is the tension in the cord?

32. Two masses of 100 gms. each are hung by a flexible cord over a frictionless pulley. A mass of 10 gms. is placed upon one of the 100 gm. masses. Required the acceleration of the system and the tension in the cord.

33. Masses of 938 and 762 gms. respectively, are hung

by a flexible cord over a frictionless pulley. How far must the masses move in order to acquire a velocity of 325 cms. per sec.?

34. A body slides down a smooth plane 326 cms. long, inclined at an angle of 45° to the horizon. Find the time of descent and the velocity with which it reaches the bottom.

35. A mass of 200 grams is constrained to move in a circle of 600 cms. radius with a velocity of 240 cms. per sec. What is the centripetal force and the period of revolution?

36. The distance of the moon from the earth is 3.84×10^{10} cm., and the lunar month is approximately 27 days and 8 hours. What is the acceleration due to the earth's attraction at the moon?

37. If a skater describe a circle of 100 ft. radius with a speed of 20 ft. per sec., find the inclination of his body from the vertical in order that he may maintain his equilibrium.

38. If the equatorial radius of the earth is 3963.3 miles, find the time of rotation necessary for a body at the equator to weigh nothing, assuming $g = 981$ cm. per sec. per sec. for the earth at rest.

39. A mass of 1 gram moves uniformly round a circle 40 cms. in the diameter at the rate of 24 revolutions a minute. Compute the force towards the center.

40. A mass of 1 gram executes simple harmonic motion with an amplitude of 4 cms. and a period of 0.5 sec. Find the force toward the center when the phase is $T/4$, $T/8$, and $T/2$, respectively.

CHAPTER IV

WORK AND ENERGY

30. Work. Work is the production of a state of motion or a state of stress, in opposition to forces tending to resist such an effect.

Examples of work are seen (*a*) in the starting of a heavy car upon a smooth level track; (*b*) in the compressing of a gas into a cylinder; (*c*) in the pumping of water into an elevated reservoir; (*d*) in the winding of a watch; (*e*) in the charging of a storage battery; (*f*) in the action of gravity upon a system composed of a wheel and axle, to which is attached a heavy weight by means of a cord wound round the axle.

When a body moves in the direction of the force acting upon it, the force is said to do work upon the body in giving it motion, as in the case of a freely falling body; if the motion be in opposition to the force, work is said to be done upon the body *against* the force, in giving the body a change in condition, or putting it in a state of stress; an example of this is seen in compressing gas into a cylinder.

The measure of work done is the product of the *force* into the *distance*, or

$$W = F s. \qquad (65)$$

If the motion take place in a line inclined to the direction of the force worked against, the effective displacement is found by projection, and the work is the product of the force into the effective displacement. An example of this is the work done in moving a body up an inclined plane. Thus the force F, (Fig. 16), in working over the distance $s = AB$, is, in reality, lifting the mass through the distance CB. The effective displacement is therefore the vertical component of s, or $s \cos \phi$, and in this case the work is

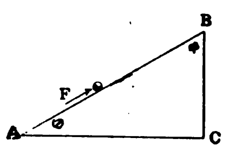


FIG. 16

$$W = F s \cos \phi \qquad (66)$$

W = Fs sin θ

W = Mg · S cos φ

where ϕ is the angle between the direction of the motion and that of the opposing force.

The C. G. S. unit of work is the *erg*. An erg is the work done by a force of one dyne acting through a distance of one centimeter; *i. e.*, unit work is done by unit force acting through unit distance. The force exerted by gravity upon a gram mass is 980 dynes. Therefore, to lift a gram mass one centimeter against gravity would require 980 ergs of work.

The dimensions of work are

$$[M\,L\,T^{-2} \times L] = [M\,L^2\,T^{-2}]$$

31. Power. It is to be noted that the expression for work, $W = F\,s$, contains no element of time. The same work is done in lifting a bag of grain to the top of a building, whether the work be done in an hour or in a month. *Power* involves the idea of the *rate at which work is done,* and may be defined as the *time rate of doing work*. Hence

$$\text{Power} = \frac{W}{T} = \frac{F\,s}{T} \qquad (67)$$

The dimensions of power are $[M\,L^2\,T^{-3}]$.

The C. G. S. unit of power is the *watt*. A watt is the power which will do 10,000,000 ergs of work in one second. The practical unit of work is the kilo-watt, or one thousand watts.

The English unit of work is the *foot-pound*; *i. e.*, the work done in lifting a pound one foot high against gravity. The corresponding unit of power is the *horse-power,* which denotes the power to do 33,000 foot-pounds of work per minute, or 550 foot-pounds per second.

One horse power is equal to 746 watts.

32. Energy. Whenever work has been done upon a system in producing a *change* either in its *motion*, its *position* or its *molecular condition,* the system has acquired the capability of doing work in turn.

Energy is the capability of doing work, possessed by a system by virtue of work having been previously done upon it. This is seen to be true of all the cases cited under Article 30. The car, once set in motion, can do work by virtue of that motion

and to the extent of that motion; the wheel and axle and the falling weight each possess energy, the one by virtue of its motion of translation, and the other by virtue of its motion of rotation; the gas compressed in the cylinder, or the coiled spring of a watch each possesses energy in the form of a *stress* or a tendency to return to a former state. The water in the reservoir possesses energy in the form of gravitational stress; the chemical elements in a storage battery, having been torn apart by the electric current, now tend to reunite and thus possess energy in the form of chemical stress.

Energy is thus seen to exist in one of two distinct forms:

(*a*) **Energy of Motion,** *or Kinetic Energy;*
(*b*) **Energy of Stress,** *or Potential Energy.*

If we do work upon a system we increase its energy, since we transfer energy from our bodies to the system. The sum total of energy in the system working and the system worked upon, is at all times a constant quantity. When work is done by one system upon another both kinds of energy are present and there is a transfer of energy from the system working to the system worked upon. Hence work may be defined as the act of transferring energy from one system to another.

It is to be noted that whenever the motion is *with* the force, the motion of the system is increased and the system gains in kinetic energy, as in the case of a freely falling body. When the motion is *against* the force, kinetic energy is changed into potential, as in the case of a body thrown vertically upward.

Strictly speaking the system working is able to transfer only its available energy to the system worked upon. The working system may possess energy in the form of heat, that is unavailable for the purpose of doing useful work. In the same way the system worked upon may receive energy from the system working in the form of heat, which escapes later, and is no longer available as energy in the system worked upon.

33. Expressions for Energy. Since by definition a system possesses energy only by virtue of work done upon it, the unit of energy is the same as the unit of work, ***the erg***. The dimensions of energy are the same as those of work, $[M L^2 T^{-2}]$.

Potential energy is stored up work, and its expression is

$$W = F\,s = M\,a\,s = \text{Potential Energy.} \tag{68}$$

To express the kinetic energy of a body in terms of its mass and velocity, we need to remember that we have a force F, acting through a space s, and the expression for kinetic energy may be deduced in a number of ways. Thus if a mass m, be acted upon by a force F, for a time t, during which it receives an acceleration a, it will pass over a space $s = \frac{1}{2}\,a\,t^2$, and acquire a velocity $v = at$. Then the work done by the force will be

$$W = F\,s = F \cdot \frac{a\,t^2}{2} = \frac{m\,a^2\,t^2}{2} = \frac{m\,v^2}{2} \tag{69}$$

or

$$\left(\text{Kinetic Energy} = \frac{m\,v^2}{2} \right) \tag{70}$$

Again since

$$v^2 = 2\,a\,s$$

we may write

$$m\,v^2 = 2\,m\,a\,s \tag{71}$$

or

$$\frac{m\,v^2}{2} = m\,a\,s = F\,s = \text{Kinetic Energy}$$

Finally we may deduce the same expression from fundamental definitions. Thus, the impulse is equal to the force into the time during which the force acts, or

$$\text{Impulse} = F\,t = m\,v - m\,v_0 \tag{72}$$

also the mean velocity $\overline{v}$, is

$$\overline{v} = \frac{s}{t} = \frac{v + v_0}{2} \tag{73}$$

whence, combining (72) and (73),

$$F\,s = \frac{m\,v^2}{2} - \frac{m\,v_0^2}{2} \tag{74}$$

This last expression states that *the work* is equal to the *change in kinetic energy* produced, and is a more general expression for the work done upon a body. By setting $v_0 = 0$, we have K. E. $= \frac{m\,v^2}{2}$ as before.

34. Transformations of Energy. Transformations of energy occur on every hand. An excellent example is seen in the motion of a pendulum bob. At the highest point of the swing its energy is all potential, at the lowest point it is all kinetic; at intermediate points it is partly kinetic and partly potential. Were it not for the slight loss of energy in overcoming the resistance of the air and the friction of the cord, this transformation would go on forever.

Consider also the transformations of energy presented in the consumption of coal in the furnace of a steam boiler. The coal supply of the world, represents the largest available source of potential energy and is simply the stored up sunshine of geologic ages. When the coal is burned in the furnace this energy becomes kinetic in the form of heat, it appears as kinetic energy in the molecular motions of the water particles in steam, and as potential energy in the stress of the steam pressure on the boiler. In the steam engine the energy becomes the kinetic energy of the moving masses of the machinery and belts, which in turn may be transformed into light, motion, electric current or chemical stress, and finally into heat again.

The transformation of energy from the potential to the kinetic form is *always a perfect one,* in that all the potential energy appears as kinetic. On the other hand the transformation from kinetic to potential is *never perfect*; some of the energy escapes as diffused heat and thus becomes unavailable for the purpose of doing useful work.

Potential energy tends to become a minimum. If in any system, any one of the stresses acting be removed, a redistribution of the energy occurs and the potential energy diminishes while the kinetic energy increases. An example of this is seen in the bursting of a reservoir full of water. The reaction of the restraining wall being removed, the water is carried down hill by the force of gravity with increasing velocity. The kinetic energy is increased at the expense of the potential. Other illustrations are seen in the bursting of a soap bubble, in the concentration of a dew drop into a sphere, and in the position assumed by any body free to move into a new position of equilibrium. The result in all cases is that the system is at rest only when the potential energy is as small as possible.

35. Conservation of Energy. Throughout all the various transformations of energy it is to be noted that no body or system of bodies can acquire energy save at the expense of energy possessed by some other system. Hence we may say that work is the process by which one system transfers its energy to another, and it seems certain from the most careful experiments, that *the amount of energy lost by the one system is the exact equivalent of that gained by the other.* This means that no machine or combination of machines can ever be made to return more energy than is given to it. Perpetual motion is a delusion. Physically it is impossible to get something for nothing. Everything must be paid for in terms of energy.

Not only is it impossible to make a machine that will create energy, no machine will ever return all the energy put into it. Owing to friction between the parts of the machine, some energy is transformed into heat and rendered unavailable for doing useful work. It is not lost, since energy, like matter, is indestructible.

The doctrine of the conservation of energy states that *in a system so situated that it neither loses energy from within nor gains energy from without, the amount of energy is constant.* No energy can be created, none can be destroyed.

PROBLEMS.

1. How many ergs of work will be done by a force of 48 dynes acting through 24 centimeters?

2. What work will be required to lift 10 kilograms of water from a well 12.5 meters deep?

3. The lower end of a ladder 16 m. long stands on the ground at a distance of 2.73 m. from a wall against which the upper end rests. How much work will be done in carrying 30 kilos up the ladder?

4. The diameter of the cylinder of a steam engine is 18 in. and its length is 24 in. What work in foot pounds will be done at each stroke of the piston if the average pressure of the steam is 110 lbs. per sq. in.?

5. If the above engine make 100 strokes per minute, calculate the horse power it will develop.

6. A shot of mass 2 kilos moving with a velocity of 200

meters per second, is just able to pierce a plank 4 cm. thick. What velocity is required to pierce a plank 12 cm. thick?

7. A stone of mass 5 kilos is thrown vertically upward with a velocity of 25 meters per second. Find its kinetic energy at the end of two seconds.

8. A bullet of 100 grams mass is discharged from a gun of mass 3 kilos, with a velocity of 400 meters per second. Compare the kinetic energies of bullet and of gun.

9. A ball of 25 kilos mass moves with a velocity 4 meters per second. Compute its kinetic energy.

10. A man whose mass is 160 lbs. carries a hod and mortar of mass 75 lbs. from the ground to a scaffold 24 feet high, every ten minutes. At what rate is he working?

11. A stand-pipe 20 meters high and 4 meters in diameter is to be filled with water from a lake 8 meters below the base of the stand-pipe. How long will it take a 10 H. P. engine to fill it?

12. A 100 gram mass is suspended from a spring balance which is carried in a balloon; what will be its apparent mass as shown by the index (*a*) when the balloon is ascending with a uniform acceleration of 240 cm. per sec. per sec.; (*b*) when it is descending with an acceleration of 900 cm. per sec. per sec.?

13. Two unequal masses are attached to the ends of a cord passing over a smooth peg; find the ratio between them in order that they may move through 500 cms. in two seconds, starting from rest.

14. The upper end of a smooth straight wire of length 100 ft., is attached to a pole 48 ft. high. A bead is allowed to slip along the wire from top to bottom. Find its velocity on reaching the bottom. Also time of descent.

15. A ball thrown up is caught by the thrower 7 seconds afterwards. How high did it go, and with what speed was it thrown? How far below its highest point was it 4 seconds after the start?

16. The mass of a pendulum bob is 100 grams, and the string is one meter long. The bob is held so that the string is horizontal, and then allowed to fall: find its kinetic energy when the string makes an angle of 30° with the vertical.

CHAPTER V

MECHANICS OF A RIGID BODY

36. Motion of a Rigid Body. A rigid body is one that suffers no change of form as a result of the forces acting upon it. When force is applied to a rigid body that is free to move, the body will acquire motion of translation or of rotation or of both together. The motion of translation imparted to the body is fully accounted for by the equation $F = m\,a$, which shows that the acceleration imparted to the body will vary directly as the force and inversely as the mass of the body. It is now necessary to investigate the conditions attending the rotation of a body as produced by one or more forces.

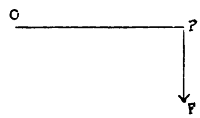


FIG. 17.

The effect of a force in producing rotation in a body about any point is termed *the moment of the force about that point.* The moment of a force about a point is the product of the force times the lever arm, or it is the product of the force into the perpendicular distance from the point to the line of action of the force. Thus in Fig. 17, the moment of the force about the point O, is $PF \times OP$. Moments are termed positive or negative according as they tend to produce rotation in the counter clock-wise, or clock-wise direction.

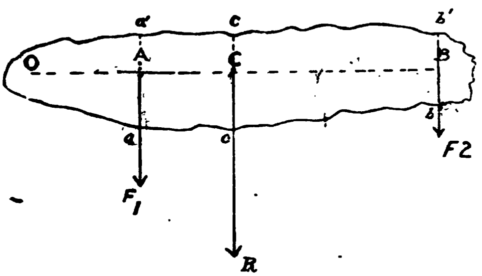


FIG. 18.

37. Resultant of Two Parallel Forces. Let F_1 and F_2 (Fig. 18), be two parallel forces applied to a rigid body in the directions $a'a$ and $b'b$; it is required to find the point of appli-

cation C, of the resultant R, such that the latter shall have the same effect in producing rotation about any point O, as the combined effect of F_1 and F_2, taken together. From O drop a perpendicular upon the direction of F_1, F_2 and R, cutting them in A, B, and C. Let $OA = x_1$, $OB = x_2$, and $OC = X$. Then by the conditions of the problem

$$RX = F_1x_1 + F_2x_2 \tag{75}$$

But by definition

$$R = F_1 + F_2 \tag{76}$$

whence

$$X = \frac{F_1 x_1 + F_2 x_2}{F_1 + F_2} \tag{77}$$

or

$$\frac{F_1}{F_2} = \frac{x_2 - X}{X - x_1} \tag{78}$$

This shows that the point of application of the resultant divides the line joining the points of application of the two forces into parts inversely as the forces. If the point O coincide with C, then

$$\frac{F_1}{F_2} = \frac{x_2}{-x_1} \tag{79}$$

or

$$F_2x_2 = -F_1x_1 \tag{80}$$

which means that the moments of F_1 and F_2 about C are equal and opposite, and there is therefore no tendency to rotate about this point, or the forces are in equilibrium about C.

If F_1 be equal and opposite to F_2, then the value for X, from (77) becomes infinite, or the solution fails for the case of *forces equal* and *oppositely directed*. This simply means that *there is no point* at which a single force can be applied so as to produce equilibrium for this system. Such a system of forces is called a *couple*. *Two equal, parallel, oppositely directed forces constitute a couple*, and if applied to the adjacent parts of a rigid body tend to produce rotation. The moment of a couple is obviously the product of one of the forces into the perpendicular distance between them.

The only way by which a couple can be equilibrated is by means of another couple, equal and oppositely directed.

38. Center of Inertia. If we take a body, as a long thin rod (Fig. 19), then every particle of matter in the rod is subject to the force of gravity, and the directions of these forces may be considered as sensibly parallel. If we take moments about any point O, situated anywhere, either inside or outside the body, we find for X, the acting distance of the resultant R,

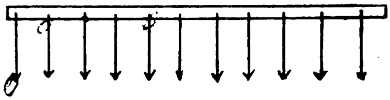


FIG. 19.

$$X = \frac{m_1 g\, x_1 + m_2 g\, x_2 + m_3 g\, x_3 + \ldots\ldots m_n g\, x_n}{m_1 g + m_2 g + m_3 g + \ldots\ldots m_n g} \quad (81)$$

or

$$X = \frac{\Sigma\, mgx}{\Sigma\, mg} = \frac{\Sigma\, mx}{\Sigma\, m} \quad (82)$$

The point defined by equation (82) is termed the *center of inertia* or the *center of mass,* or the *center of gravity* of the body. It denotes the point of application of the resultant of all the forces of gravity acting upon the body. If a rigid support be placed under this point in the body, all the forces of gravity acting upon the body will be in equilibrium, since the sum of their moments about this point is equal to zero.

This general principle may be applied to a few simple cases.

(*a*) The center of inertia of a line is at its middle point.

(*b*) The center of inertia of any lamina having an axis of symmetry, lies on this axis. If the lamina have two axes of symmetry, then the center of inertia of the lamina lies at their intersection.

(*c*) The center of inertia of a triangle lies at the intersection of the median lines of the triangle.

(*d*) The center of inertia of any polygon may be determined by dividing the polygon into triangles and considering the weight of the triangles as applied at their respective centers of inertia.

(*e*) The center of inertia of a triangular pyramid may be shown to be three fourths of the distance from the apex to the center of inertia of the triangular base of the pyramid.

39. Conditions of Equilibrium. Since the motion of a rigid body is either a motion of translation or of rotation, or a

combination of the two, it follows that for the body to be in equilibrium, there should be no tendency to produce either translation or rotation. Hence the conditions for equilibrium are

$$\Sigma F x = 0 \qquad (83)$$

$$\Sigma F = 0 \qquad (84)$$

where moments and forces are to be taken with regard to their proper signs. The origin or center about which the moments are to be taken may be situated anywhere, as its position is of no importance.

40. Stability of Bodies. The principle of moments may be applied to the determination of the stability of bodies. A body is said to be *stable, or in stable equilibrium,* when it is necessary to do work upon it in order to displace it. In such a case the center of inertia must be raised, if the body is to be displaced, and the body, if disturbed, tends to return to its original position. If, on the other hand, the body on being disturbed has its center of inertia lowered, or tends to fall farther from its original position, it is said to be in *unstable equilibrium* A book lying on its side on a smooth table is in stable equilibrium. If the book be set upon edge or upon end it is in relatively unstable equilibrium. A ball or a cone lying on its side upon a smooth horizontal surface, is said to be in *neutral or indifferent* equilibrium.

In the case of a body of regular outline, as a parallelopiped or a cylinder, the stability of the body in any position is measured by the moment of the restoring force. Thus in Fig. 20, let $ABCD$ represent a rigid body, and let G be its center of inertia, distant h above the base $DC = b$. If we attempt to overturn the body about either C or D the restoring moment M, tending to right the body, is

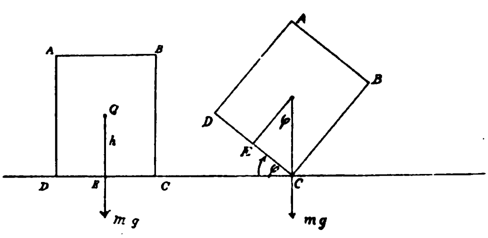


FIG. 20.

$$M = mg \times EC = mg \cdot \frac{b}{2} \qquad (85)$$

This moment decreases as the body is tipped over, and vanishes when the body has been tipped through the angle ϕ, such that the line of action of the force passes through the axis of rotation at C. Hence the stability of the body is a function of ϕ, the angle of tip necessary to overturn it. From the figure

$$\tan \phi = \frac{b}{2h} \tag{86}$$

From this it appears that the stability of the body varies directly as the base b, and inversely as the hight h, of the center of inertia above the base. A pyramid is a very stable structure, because of its broad base and also because its center of inertia lies low in the body.

The stability of quadrupeds is great, on account of the broad base, represented by the area enclosed by the four feet, and because the center of inertia is almost vertically over the center of the base. For the kangaroo, though an apparent biped, the tail acts as a third leg and the base is correspondingly increased. In man the base of support is small and the center of inertia is high, and the stability of the upright body is not great. In corpulent persons the effort to keep the center of gravity over the base leads to a swaying or rolling motion in walking, which is seen in an exaggerated form in the waddling walk of a duck or a goose.

41. Machines. A machine is any device for so transforming or transferring energy as to enable man to employ the forces of nature for doing useful work. In all cases a certain amount of energy is applied *to* the machine and in return a certain amount of useful work is done *by* the machine. An ideally perfect machine would return all the energy applied to it. In actual practice a portion of the applied energy is always expended in overcoming friction, and is thus frittered away as heat. Notwithstanding this loss, machines are employed for the purpose of applying to better advantage the forces at our command.

Since Energy = Work = Force × Space, it is evident that with a given amount of energy, the force may be *increased* exactly in the ratio that the distance through which it acts is *decreased*. Thus if P be the force applied to the machine, and s the distance through which it moves, W the force exerted by

the machine, and s' the distance through which the machine works, then, if we neglect friction we have

$$Ps = Ws' \qquad (87)$$

or

$$\frac{W}{P} = \frac{s}{s'} \qquad (88)$$

The ratio W/P between the force exerted and the force applied is called the *mechanical advantage* of the machine.

Newton described six elementary machines to which all others may be reduced. These are the *lever,* the *pulley,* the *inclined plane,* the *wheel and axle,* the *wedge* and the *screw.* These may be reduced to three, since the wheel and axle is but a modified lever, and the screw and the wedge are but modifications of the inclined plane. In all cases, however, the law of machines holds good, viz: *the force applied, multiplied into its acting distance, is equal to the force exterted multiplied into its acting distance.* A few examples will illustrate.

42. Simple Machines. *The Lever.* The lever is a rigid bar turning freely about a fixed point called the fulcrum, which is to be considered as the center about which the moments of the forces P and W are to be taken.

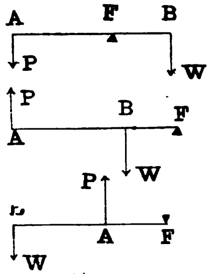


FIG. 21.

For equilibrium these moments must be equal and opposite, or

$$\Sigma Fx = 0$$

According to the relative position of the fulcrum and the points of application of the forces P and $W,$ levers are classified as follows: (Fig. 21.)

(*a*) *Levers of the first class*; fulcrum between the power and the weight. Examples, a crow-bar, a pair of scissors, or a pair of forceps.

(*b*) *Levers of the second class*; weight between the fulcrum and the power. Examples, a pair of nut-crackers, a wheel barrow, or a door swung by its outer edge.

(*c*) *Levers of the third class*; power between the weight and the fulcrum. Examples, a pair of sugar tongs, or a pair of tweezers; a door swung by its inner edge; the human forearm.

The pulley. A pulley is a small wheel turning freely about an axis, and having a grooved rim for the reception of a cord. A fixed pulley serves simply to change the direction of the applied force, but offers no mechanical advantage. It is a lever with equal arms. A movable pulley, (Fig. 22), permits of giving different velocities to P and W, and its mechanical advantage is proportional to the number of strands of cord supporting the weight.

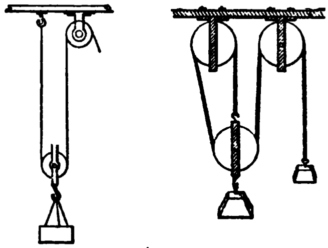


FIG. 22.

The Inclined Plane. In the inclined plane we have a plane surface inclined at an angle θ to the horizon. Three cases are possible:

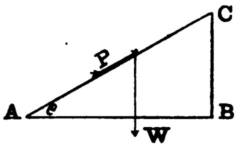


FIG. 23.

(*a*) *Power acts parallel to the plane.* In this case the force P, (Fig. 23), tending to produce motion *up* the plane is balanced by the component of the force of gravity *down* the plane, or for equilibrium

$$\Sigma F = 0$$

Hence we have

$$P = M g (\cos 90^\circ - \theta) = M g \sin \theta \qquad (89)$$

If we represent the length of the plane by l, the hight by h, and the base by b, we have, on substituting the value of $\sin \theta$ in the above expression,

$$P = M g \frac{h}{l}$$

or

$$P l = M g h \qquad (90)$$

This shows that the work done in pushing the mass M up the plane is the same as that required to lift it through the vertical hight h.

(*b*) *Power acts parallel to the base of the plane.* In this case the power P, (Fig. 24), furnishes a component up the plane which must be balanced by the component of the weight down the plane, or

~~24), furnishes a component up the plane~~

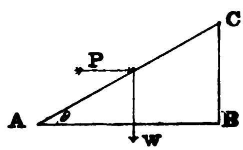


FIG. 24.

$$P \cos \theta = M g \sin \theta \qquad (91)$$

whence

$$P\frac{b}{l} = Mg\frac{h}{l}$$

or

$$Pb = Mgh \tag{92}$$

and again the work done in the two cases is equal.

(*c*) *Power acts at an angle ϕ with the plane.* In this case the equation of equilibrium is obtained as before by setting the components parallel to the plane equal to each other, or

$$P\cos\phi = Mg\sin\theta \tag{93}$$

The inclined plane permits of a small force being employed to do work by moving through a correspondingly greater distance. The winding curves of a railway as it makes its way up the mountain side, illustrate the application of this principle.

43. Friction. Friction is the resistance offered to the motion of one portion of matter upon another. It seems to be due partly to the molecular attraction between the surfaces in contact, and partly to the nature and condition of those surfaces. Even the most highly polished surface is covered with minute scratches and projections which oppose the motion of a similar surface over it. Friction always acts tangentially to the surfaces in contact and always opposes the motion.

Friction is of two kinds: (*a*) *static friction,* or the resistance offered to a force tending to start one body to slide uniformly upon another; (*b*) *kinetic friction,* or the force necessary to keep the body sliding in uniform motion on a horizontal surface. Static friction is always greater than kinetic friction and varies from zero up to a maximum value, but is never more than that required to prevent motion.

The following facts concerning friction seem well established by experiment: (*a*) The friction depends upon the nature of the substances but is independent of the area of the surfaces in contact.

(*b*) The friction is independent of the speed of the motion unless it be very small, when it rises to the maximum or static friction, as the speed falls to zero.

(*c*) The friction is proportional to the force with which the surfaces are pressed together: *i. e.*, it is that force times some

factor which is always less than unity. This factor is called the *coefficient of friction.*

44. The Balance. The balance is an instrument for the comparison of masses. It is a lever of the first class, having equal arms. In its simplest form it consists of a light, strong beam, supported at its center by a knife-edge resting upon an agate plate, and carrying upon knife-edges at its ends two pans for the reception of the masses to be compared. When disturbed the system oscillates about its position of equilibrium, to which it finally returns. When masses are placed in the pans, it is evident that the original position of equilibrium will be resumed only when the moments of the forces of gravity acting upon these masses are equal. If we assume the arms of the balance to be equal, we may set the masses equal to each other when their moments have been shown to be equal; *i. e.,* when the balance resumes its original position of equilibrium.

In case the arms of the balance are not of equal length, the true weight of the body may yet be determined by the method of double weighing. Let the arms of the balance be l and l_1, and let the true weight of the body be W. Suppose the body when hung from the arm l, to be balanced by a weight W_1 on l_1, and when hung from the arm l_1, to be balanced by a weight W_2 on l. Then by the principle of moments we have for the first case

$$Wl = W_1l_1 \tag{94}$$

and for the second

$$W_2l = Wl_1 \tag{95}$$

Dividing the first equation by the second we have

$$\frac{W}{W_2} = \frac{W_1}{W} \tag{96}$$

or

$$W^2 = W_1W_2 \tag{97}$$

whence

$$W = \sqrt{W_1 W_2} \tag{98}$$

From this it appears that the true weight of a body may be found upon a balance of unequal arms by weighing first in one pan and then in the other and taking the square root of the product. If W_1 and W_2 are not greatly different from each other.

the arithmetical mean of the two weights is usually sufficient unless extreme accuracy is required.

45. Sensibility of the Balance. The delicacy of the balance is determined by its sensibility. This depends upon a number of considerations and is expressed in terms of the angular displacement of the beam for a small difference of weight p, usually 1 milligram, in the pans.

The sensibility of a balance is determined as follows: Let the points A, B, C (Fig. 25), represent the knife-edges of the two pans and the beam, and suppose these points to lie in a horizontal line. Let the center of gravity of the system be at G, distant r from the center of suspension C, and let the weight of the system be w. For two equal weights P and P', placed in the pans, the system is in equilibrium. If an additional small weight p, be placed in the right hand pan, the system is displaced through the angle θ, and finally comes to rest in the position $A'CB'$. In this position the restoring moment due to w, and the disturbing moment due to p, must be equal and opposite. Then

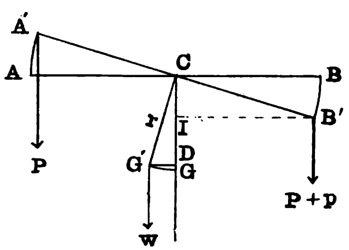


FIG. 25.

$$w \times G'D = p \times B'I \tag{99}$$

Substituting for $B'I$ and $G'D$ their values in terms of θ, we have

$$w\, r \sin\theta = p\, l \cos\theta \tag{100}$$

whence

$$\frac{\tan\theta}{p} = \frac{l}{w\, r} \tag{101}$$

The quantity $\frac{\tan\theta}{p}$ is termed *the sensibility of the balance* and is seen to vary directly as the length of the beam, and inversely as the weight of the system, and as the distance of the center of gravity of the system, from the point of support.

In order, therefore, for the balance to be sensitive, the beam should be long and light and the distance r, between the center of gravity of the system and the center of support should be small. Practical considerations, however, set a limit to each

of these theoretical suggestions. The beam must neither be so long nor so light as to render it liable to bend under the load it is to carry, else the sensibility would be diminished; also if the distance r be made too small the system becomes unstable and the time of vibration becomes inconveniently long.

46. Moment of Inertia. It is now necessary to investigate the expression for the kinetic energy of a rotating body. Suppose a rigid body to rotate uniformly about a line through its center of gravity, O, (Fig. 26), and let us call this line the axis of rotation. Also let ω denote the angle swept out in unit time. Then two particles of masses m_1 and m_2 situated at distances r_1 and r_2, from this axis, describe circles of circumferences $2\pi r_1$ and $2\pi r_2$ respectively, *in the same time, T.* The *linear velocities* v_1 and v_2, of the particles are consequently $\frac{2\pi r_1}{T}$ and $\frac{2\pi r_2}{T}$. But since $\frac{2\pi}{T} = \omega$, we have

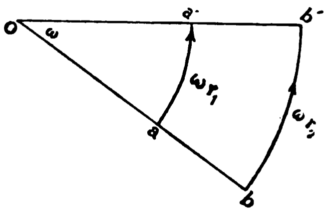


FIG. 26.

$$v_1 = \omega r_1$$
$$v_2 = \omega r_2$$

or in general, the *linear velocity* of any point revolving about an axis, is equal to the *angular velocity times the radius*; i. e.,

$$v = \omega r \tag{102}$$

Again, the kinetic energy of the two particles m_1 and m_2 will be

$$\frac{m_1 v_1^2}{2} = \frac{m_1 \omega^2 r_1^2}{2}$$

and (103)

$$\frac{m_2 v_2^2}{2} = \frac{m_2 \omega^2 r_2^2}{2}$$

If we suppose the rotating body to be made up of an indefinite number of such particles, then the kinetic energy of the body will be the sum of the kinetic energies of all its particles, or

$$K.\,E. = \frac{m_1 \omega^2 r_1^2}{2} + \frac{m_2 \omega^2 r_2^2}{2} + \frac{m_3 \omega^2 r_3^2}{2} + \dots \frac{m_n \omega^2 r_n^2}{2} \tag{104}$$

or, since ω is constant for the entire body,

$$K.E. = \frac{\omega^2}{2} \Sigma m r^2 \tag{105}$$

The expression, $\Sigma m r^2$ is called the *moment of inertia* of the body, and is designated by the letter I. It denotes the sum of the products, obtained by multiplying the mass of each individual particle by the square of its distance from the axis of rotation. This quantity, $\Sigma m r^2$, has a perfectly definite, positive, numerical value for a definite body, rotating about a definite axis. It will be observed that I depends, not only upon the *mass* of the body, but much more upon the *manner in which that mass is distributed with regard to the axis of rotation.* For every body it is possible to find a radius K, such that the mass of the body M, multiplied by the square of this radius is equal to the moment of inertia, or

$$M K^2 = \Sigma m r^2 \tag{106}$$

Such a radius is called the *radius of gyration.*

The dimensions of a moment of inertia are $[M L^2]$ and the unit is gm. cm².

For convenience of reference the following table of moments of inertia is appended, where M denotes the mass of the body and I the required moment of inertia.

MOMENTS OF INERTIA. (FORMULÆ).

Uniform thin Rod, axis through middle, length $= l$,

$$I = M \frac{l^2}{12} \tag{107}$$

Uniform thin Rod, axis at one end, length $= l$

$$I = M \frac{l^2}{3} \tag{108}$$

Rectangular Lamina axis through center and *parallel* to one side, length, of sides a and b, side b bisected,

$$I = M \frac{b^2}{12} \tag{109}$$

Rectangular Lamina, axis through center and *perpendicular* to the plane, sides a and b,

$$I = M\frac{a^2 + b^2}{12} \tag{110}$$

Circular Plate, axis through center *perpendicular* to the plate, radius $= r$

$$I = M\frac{r^2}{2} \tag{111}$$

Circular Plate, axis any diameter, radius $= r$

$$I = M\frac{r^2}{4} \tag{112}$$

Circular Ring, axis through center perpendicular to plane of ring, outer radius $= R$, inner radius $= r$

$$I = M\frac{(R^2 + r^2)}{2} \tag{113}$$

Circular Ring, axis any diameter, radii as before

$$I = M\frac{(R^2 + r^2)}{4} \tag{114}$$

Moment of Inertia about an axis parallel to an axis through center of gravity of body, and distant from same by a distance a, moment of inertia about axis through center of gravity $= I_0$,

$$I = I_0 + M\,a^2 \tag{115}$$

47. Moment of Inertia and Angular Acceleration. If α be the angular acceleration, then

$$\alpha = \frac{\omega - \omega_0}{t} \tag{116}$$

also the linear acceleration

$$a = \frac{v - v_0}{t} = \frac{\omega\,r - \omega_0\,r}{t} = \alpha r$$

or the linear acceleration is equal to the angular acceleration times the radius

$$a = \alpha r \tag{117}$$

If now we consider a particle of mass m, at a distance r from the axis of rotation, then the force necessary to give it a linear acceleration a, is

$$F = m\,a = m\alpha r \tag{118}$$

and the moment of the force is $m\,\alpha\, r \times r$ since the force acts tangentially to the circle described by m, or the moment *for a single particle* is $m\,\alpha\, r^2$.

The moment of the force necessary to give the entire body the same acceleration is $\Sigma\, m\,\alpha\, r^2 = \alpha\, \Sigma\, m\, r^2$. If we set α equal to unity, then ***the moment of the force necessary to produce unit angular acceleration is*** $\Sigma\, m\, r^2 = I$, ***the moment of inertia.*** Hence from this point of view we may define the moment of inertia of a rigid body, $\Sigma\, m\, r^2$, as the moment of a couple which confers upon the body ***unit angular acceleration.***

48. Kinetic Energy of Rotation. We have now derived two distinct formulæ for kinetic energy.

(*a*) *K. E. of translation* $= \frac{1}{2} M v^2$ (119)

(*b*) *K. E. of rotation* $= \frac{1}{2} I \omega^2$ (120)

Our comparing these formulæ for kinetic energy, we note from their symmetry that the angular velocity ω, of a rotating body, corresponds to the linear velocity v, of a moving body; *also that the moment of inertia I, of the rotating body, corresponds to the mass M, of the moving body.* This relation is most important, and finds constant application in mechanics. The fly-wheel of an engine acts as a reservoir of kinetic energy, not only on account of the mass, but still more on account of the distribution of its mass, since this is largely in the rim, thus making its moment of inertia as large as possible.

Again it should be observed that a system may possess kinetic energy both of translation and of rotation at the same time. Thus the wheel of a bicycle when in motion possesses kinetic energy due to its motion of translation equal to $\frac{1}{2}\, m\, v^2$, and also due to its rotation amounting to $\frac{1}{2}\, I\, \omega^2$. This fact has been ingeniously employed in the manufacture of steel shells for modern, long distance, rifled cannon. The flying cylindrical shell possesses both energy of translation and energy of spin around its longer axis, due to the twisting motion imparted to it by the rifling of the gun. The striking end of the shell is now made of the hardest steel, and furnished with a screw point, so that the instant it strikes a target, the kinetic energy of rotation causes it to bore in like an auger, thus increasing its destructive action many fold.

49. Ideal Simple Pendulum. The ideal simple pendulum may be defined as a heavy material particle, suspended by a weightless thread. As an approximation to this ideal we may use a lead ball supported by a fine silk cord. It is required to find an expression for the time of vibration of this pendulum. Let the pendulum of mass m, be supported by a cord of length $AB = l$, (Fig. 27); when drawn aside to the position B, it forms an angle $BAN = \theta$, with the vertical and the component of $BG = mg$ which is effective in producing motion, is $m\,g \cos(90^\circ - \theta) = m\,g \sin\theta$. Let $NB = x$, and let f_x be the acceleration toward the position of rest, then mf_x is the force urging the body toward N, or

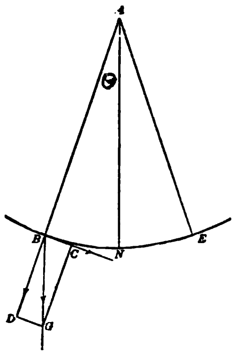


FIG. 27.

$$mf_x = -m\,g \sin\theta \qquad (121)$$

whence

$$f_x = -g \sin\theta \qquad (122)$$

also

$$\theta = \frac{x}{l} \qquad (123)$$

If θ be small, we may set

$$\sin\theta = \theta = \frac{x}{l}$$

therefore

$$f_x = -\frac{g}{l} \cdot x \qquad (124)$$

Now since g and l are constants, it is clear that within the limits where we may write $\sin\theta = \theta = \tan\theta$, the *acceleration is proportional to the displacement*, and the motion of the pendulum will be *simple harmonic motion*. But from equations (60), we have

$$f_x = -\frac{4\,\pi^2}{T^2}\,x$$

therefore

$$\frac{4\,\pi^2\,x}{T^2} = \frac{g}{l}\,x \qquad (125)$$

or

$$T = 2\pi\sqrt{\frac{l}{g}} \qquad (126)$$

This is the time of a *complete vibration*; for a half vibration

$$T = \pi\sqrt{\frac{l}{g}} \qquad (127)$$

To find the length of the seconds pendulum, where

$$g = 980\,\frac{cm.}{sec.^2},\ \text{set } T = 1,\ \text{or } 1 = \pi\sqrt{\frac{l}{980}}$$

from which

$$l = \frac{980}{\pi^2} = 99.3\ cm.$$

50. Compound or Physical Pendulum. Any body suspended so as to swing freely about an axis, forms a compound or physical pendulum. In such a pendulum the mass of the body may be considered as concentrated at its center of gravity. Suppose the heavy body, (Fig. 28), be drawn aside through an angle θ, then the moment of the restoring force is $Mgh \sin\theta$, where h is the distance from the point of suspension to the center of gravity of the pendulum. This same moment tends to produce an angular acceleration a, and from Article 47, we have seen that such a moment is equal to $a\ \Sigma\ m\ r^2$, hence equating these two values for the moment of the force Mg upon the pendulum, we have

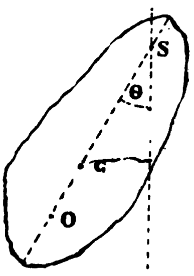


FIG. 28.

$$a\ \Sigma\ m\ r^2 = M\ g\ h \sin\theta \qquad (128)$$

whence

$$\frac{\Sigma\ m\ r^2}{M\ h} = \frac{g \sin\theta}{a} \qquad (129)$$

Now $\frac{g \sin\theta}{a}$ is a linear acceleration divided by an angular acceleration, and from Article 47, we have seen that the quotient must be *a length*; hence $\frac{\Sigma m r^2}{M h}$ is also a length; that is, it is the length of an equivalent simple pendulum, which will vibrate in the same time as the body. If we let

$$l = \frac{\Sigma\ m\ r^2}{M\ h} \qquad (130)$$

equal the length of this simple pendulum, we may write the expression for the time of vibration of the body at once.

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{\Sigma m r^2}{M h g}} \qquad (131)$$

This is a general expression for the time of vibration of any heavy body when disturbed, about its position of equilibrium.

The length of a compound pendulum may be determined experimentally by suspending near it a simple pendulum and altering the length of the latter until the two vibrate in the same time. The length of the simple pendulum so found is

$$l = \frac{\Sigma m r^2}{M h}$$

If now, we lay off this length upon the compound pendulum, measuring *from* the center of suspension *through* the center of gravity, we find the *center of oscillation* O; that is, the center of oscillation is distant from the center of suspension by the length l, as defined by equation (130), and the pendulum behaves as if all the mass were located at this point.

It may be also shown that the center of oscillation and the center of suspension are interchangeable; or in other words, for every point of suspension S, there exists a conjugate point O, such that the pendulum will vibrate in the same time, whether suspended from S, or from O.

The center of oscillation is also termed the *center of percussion*, since, if the pendulum be struck a blow at this point, it will move forward in its arc, with no jerk or jar upon the point S, as if the whole mass had received the blow. At any other point the effect of a blow would be to start the pendulum to swinging, and also to jerk it from its support owing to the tendency to rotate about some axis other than the point of support. This is well illustrated in the case of a base ball bat, which is to be considered as a compound pendulum. If the ball be received upon the bat at the center of percussion, no jarring of the hands is experienced by the striker. If the ball strike the bat at any other point, either too near the end or too near the hands, the jarring effect is very painful, and the bat is usually broken as well.

PROBLEMS

1. A weight of 350 kilos is carried by a bar, AB, 4.7 meters long, supported at the ends. If the distance of the weight from the end A be 2 meters, what is the weight borne by each support?

2. A uniform lever 8.2 feet long weighs 15 pounds. If 26 pounds be hung from one end and the bar be sustained by a support 2.6 feet from this end, what weight must be hung from the other end for the system to be in equilibrium?

3. It is found that a lever cut from a uniform bar weighing 4.2 pounds to the foot, balances at a distance of 2.3 ft. from one end when weighted at this end with 5.4 pounds. What is the length of the bar?

4. When a lever, AB, is supported at its center of gravity, it is found that a weight, W, hung at A will balance 2.5 pounds at B; but when W is hung at B it requires a weight of 19 pounds at A to keep it in equilibrium. What is the weight W?

5. Find the magnitude and point of application of the resultant of two parallel forces, in the same direction, equal respectively, to 25 and 42, when their lines of action are 3.4 feet apart.

6. A meter scale, (of which the mass may be neglected) has suspended from the 10 cm. mark and from the 90 cm. mark masses of 2 and 9 kilos respectively. Where must the scale be supported for equilibrium?

7. A lever is 20 cm. long and its mass 15 gms. Where must the fulcrum be placed in order that a mass of 10 gms. at one end may just balance a mass of 16 gms. at the other end?

8. A uniform, thin rod one meter long, whose mass may be neglected carries four masses of 4, 7, 10 and 22 grams. The 4 gram mass is at one end and the 22 grams at the other, while the 7 and 10 gram masses are 55 and 75 cms., from the 4 gram mass. Find the center of gravity of the system.

9. From a square plate of uniform thickness, one triangle formed by the intersection of the diagonals has been removed; find the center of gravity of the remainder.

10. Masses of 7, 9 and 11 kilos are placed at the points A, B and C of a right triangle, right angled at C, of which the base

AC, is 10 cm. and the altitude *BC*, is 15 cm. and whose mass may be neglected. Find the center of gravity of the system.

11. Find the center of gravity of the complete figure formed of a right triangle, and of the three squares described upon its sides.

12. A wheel of mass 100 pounds is attached to end of a straight uniform axle 4 feet long and of mass 50 pounds. Find the center of gravity of the system.

13. In a system of one fixed and one movable pulley, (Fig. 22), what weight can be lifted by a force of 20 pounds weight, neglecting friction and mass of pulley?

14. In a system of two fixed and one movable pulley, (Fig. 22), what mass at end of cord will equilibrate 270 kilos attached to movable pulley, neglecting friction and mass of pulley as before?

15. What force acting horizontally can keep a mass of 16 kilos at rest on a smooth inclined plane, whose height is 3 meters and base 4 meters?

16. Find pressure on the plane mentioned in problem 15.

17. A body of mass 25 kilos rests upon an inclined plane, which makes an angle of 30° with the horizon; find the force necessary to produce equilibrium, (*a*) when the force acts parallel to plane, (*b*) when parallel to base.

18. A boy who can exert a force equal to 50 lbs. weight, wishes to roll a barrel of mass 200 lbs. into a window four feet high. How long a plank is required?

19. If a mass weigh 29.62 gms., in one pan of a balance and 28.71 gms. in the other, what is its true weight?

20. A mass of 50 pounds moving with a velocity of 10 ft. per second overtakes a mass of 25 pounds moving with a velocity of 6 ft. per second. If both bodies are perfectly elastic, what will be their velocities after impact?

21. If a mass of 625 gms. moving with a velocity of 786 cm. per sec. meet a mass of 164 gms. moving in the opposite direction with same speed, what will be their velocity after impact, supposing the bodies to be inelastic?

22. A grindstone of radius 20 cm. and density 2.2 gm. per

cc., is 10 cm. thick. Calculate its moment of inertia when rotated about an axis through the center.

23. If this stone rotate 5 times in 4 minutes, compute its kinetic energy.

24. Compare the moment of inertia of a meter rod of mass 150 grams, when rotated about an axis, (*a*) through one end, and (*b*) through the center, both axes being normal to the length of the rod.

25. A board 2 cm. thick and 8 cm. long by 5 cm. wide, of density 0.75 g. per cc., rotates about an axis, through its center of gravity and normal to its plane. Compute its moment of inertia.

26. A grindstone of mass 25 kilos and radius 40 cm. rolls without sliding down an inclined plane 200 cm. long and 50 cm. high. Compare its kinetic energies of rotation and of translation.

27. Find the time of vibration of a simple pendulum 20 cm. long, where g equals 980 cm. per sec., per sec.

28. What would be the value of g, if the period of a simple pendulum 97.31 cm. in length were 1.975 sec.?

29. A pendulum loses 20 sec. per day, where g equals 980.3 cm. per sec., per sec.; find its length.

30. A compound pendulum is composed of a wire 60 cm. long, whose weight may be neglected, suspended from one end, and carrying masses of 3, 5, and 7 kilos at distances of 20, 40 and 60 cm. from the point of support. Compute its length and time of double vibration.

31. A thin hoop of radius 30 cms. and mass 100 grams, hangs over a fine nail and vibrates in its own plane. Find its time of vibration.

32. A hollow cylinder rolls down a perfectly rough inclined plane in 10 minutes; find the time a uniform solid cylinder would take to roll down the same plane.

33. The weights in an Atwood's machine are 355 and 345 grams. In the *second* second (starting from rest) the heavier weight descends through 27 cm. What value does this give for g?

CHAPTER VI

ELASTICITY

51. Stress and Strain. Elasticity is that property by virtue of which matter resists the action of a force tending to change its shape or bulk, and which causes it to resume its original shape or bulk after the force is removed. If a body possess elasticity of shape it is called *a solid,* if it possess no elasticity of shape it is called *a fluid.* Any change in a body either of size or shape produced by the action of a force is called *a strain.* The force producing such a change is called *a stress,* and is *measured in dynes per unit area* of the cross section upon which the stress is exerted.

Fluids possess *perfect elasticity of bulk*; that is, they return exactly to their original bulk on removal of the compressing stress. For every solid there is a limiting distortion beyond which the body when freed from the distorting stress no longer completely regains its former shape. This is called *the limit of elasticity.* All solids do not recover their initial shape with equal promptness. In some cases this return is much retarded, especially after repeated or long continued distortion. This is commonly termed *elastic fatigue,* and is quite noticeable in metals.

52. Hooke's Law. When an elastic body is distorted within its limit of elasticity, the opposing force called out by the distortion, tending to restore the body to its original condition, is *proportional to the distortion.* This is known as *"Hooke's Law,"* and as originally stated, *"ut tensio sic vis,"* expresses the proportionality between the distortion and the restoring force. The applications of this law are very numerous, including every form of elastic reaction against strains produced by external mechanical agencies.

As will be readily seen, the application of Hooke's law involves the fundamental assumption that the force, and therefore the *acceleration* is proportional *to the distortion.* If the

original position of the system be taken as the position of rest, we have at once the characteristic condition of simple harmonic motion. Such motions are seen in the vibrations of elastic springs, of the air particles in the production of sound, in the varying charges in an electric condenser, and in compressional and torsional waves in solids.

X **53. Coefficients of Elasticity.** In general twenty-one coefficients would be necessary to express completely the elastic nature of a body. If, however, the body be isotropic these twenty-one coefficients reduce to two: the *coefficient of volume elasticity e, and the coefficient of simple rigidity, n.* The general expression for these coefficients is the quotient arising from *dividing the stress by the strain.*

In the coefficient of volume elasticity *e,* we have the stress or applied pressure *p,* divided by the compression produced, where *compression* denotes the *change in volume v, divided by the original volume V,* or

$$e=\frac{p}{\frac{v}{V}}=\frac{p}{v}V \tag{132}$$

Since all changes in volume are conceived as being very small, if we assume a volume of gas *V,* to be subjected to a change in pressure *dp,* producing a corresponding change in volume *dV,* then

$$e=\frac{dp}{dV}V \tag{133}$$

It should be observed that the expression for the strain $\frac{dV}{V}$ denotes a *dilatation if positive,* and a *compression if negative.* The coefficient *e,* however, has reference simply to the numerical magnitude of the ratio, $\frac{dp}{\frac{dV}{V}}$ and is therefore independent of the sign.

Besides the elasticity of volume, solids have, as we have seen, elasticity of shape as well. If a solid be so distorted that its shape alone is changed it is said to have undergone a *shear.* Thus if we conceive all the particles in a certain plane in a body

to be fixed, and all the remaining particles to move in planes parallel to this plane, and by amounts proportional to their distances from this plane, *such a motion is called a shear.* The stress applied to cause such a motion is called a *shearing stress,* and the coefficient of simple rigidity is the quotient obtained by dividing the shearing stress per unit area, by the shearing strain per unit length.

54. Three States of Matter. Matter presents itself in one of three states: *solid, liquid* or *gaseous.* These states are not separated by sharp lines of demarcation but shade almost insensibly into each other, the particular state which the body assumes being dependent upon the temperature and pressure to which it is subjected.

A solid is a body which resists any stress tending to change either its shape or its volume; in other words, it possesses elasticity both of shape and of bulk. It is characterized by small mobility of its molecules, and by the possession of surfaces of distinct outline on all sides.

A liquid is a body which has no elasticity of shape, or which offers no resistance to a shearing stress; it is characterized by considerable mobility of its molecules, by a distinct, free, upper surface, usually of a meniscus shape when confined in a tube, and by the existence in that free surface of a specific stress or tension, not found elsewhere in the body.

A gas is a body whose parts are held together only by the action of external stress, and which offers resistance to the *decrease* but none to the *increase* of its volume. It is characterized by perfect mobility of its molecules. It has no upper surface and no surface tension, but tends to fill all available space.

Liquids and gases when subjected to shearing stresses are unable to remain in equilibrium, and their parts move into new positions, or tend to *flow*; they are therefore classed together under the general term *fluids.* Fluids have perfect elasticity of volume but no elasticity of shape.

55. Intermediary Qualities. As has been said, the three states of matter shade gradually into each other, although the changes are more abrupt in some cases than in others. Thus while most substances are perfectly elastic under hydrostatic

pressure, no substance has yet been found to be perfectly elastic under shearing stresses producing finite strains. Owing to the varying behavior of substances under stress, a number of intermediary qualities are manifested, by virtue of which certain characteristics of the fluid state are seen to exist even in solids.

A body which breaks upon the application of slight stress is said to be *brittle.* Glass, gems and hardened steel are examples of brittle substances.

A *plastic* substance is one which has its form permanently altered by a stress exceeding a certain value. Plastic solids under pressure behave as liquids, and flow with greater or less freedom. Thus cold lead under enormous pressure is made to flow through a die, producing lead pipe.

Plastic bodies may be beaten into various forms, rolled into sheets, or drawn into wire. This is exemplified in the case of copper and platinum and in still higher degree in the case of gold. Substances which may be beaten into thin sheets are called *malleable*; those that may be drawn into wire are said to be *ductile.* Certain substances are plastic at high temperatures and brittle at ordinary temperatures. Thus glass and quartz are extremely brittle at ordinary temperatures, yet when fused they may be drawn into threads of exceeding fineness. Quartz especially, when fused may be drawn into fibers so fine as to be invisible to the naked eye, yet capable of supporting relatively large masses, and possessing almost entire freedom from elastic fatigue.

Certain organic substances rival the metals in ductility. Thus sugar and pitch at certain temperatures may be drawn into threads so fine as to float on the air, while the spider's thread is formed in some cases of a thousand separate strands. united into one.

56. Viscosity. While it is assumed that an ideal liquid would offer no resistance to a shearing stress, still it is to be noted that all known liquids do manifest a certain degree of resistance to the motion of their parts over each other. This *internal friction* or *resistance to flow*, is called *viscosity.* A liquid that flows readily as alcohol or ether is termed a *limpid* liquid as opposed to a *viscous* liquid like honey or Canada balsam.

To illustrate the viscosity of liquids, support horizontally,

a glass disk 10 cms. in diameter, by a stiff wire from its center, and lower it into a flat beaker containing several centimeters depth of glycerine, bringing the disk at least 5 cms. below the surface of the liquid. Place small bits of paper or cork upon the glycerine and rotate the disk by twisting the supporting wire. The surface is soon seen to be in rotation, showing that the rotation of the disk sets up a shearing stress which is transmitted through the liquid to the surface layers and sets them in motion also. A smooth metal disk mounted horizontally upon a whirling table and set in rapid rotation will drag along with it the adjacent layers of air, as may be shown by lowering over it a similar disk suspended by a fine cord from its center. The viscosity of the air will soon suffice to set the upper disk in rotation even at a distance of one or two centimeters.

It thus appears that a column of glycerine or even a column of air is able, by virtue of the internal friction between its molecules, to transmit a shearing stress from one end of the column to the other. In the case of a perfectly rigid column, such transmission would be instantaneous, while in the case of fluid columns there is a continual breaking down under the stress and the transmission is much slower and far less perfect.

This continued breaking down or yielding of the parts of a body under the action of a shearing stress is seen also in the case of bodies usually classed as solids. Thus shoemaker's wax, while brittle to a sudden blow, will creep slowly down an inclined plane under the action of gravity; long glass tubes, laid horizontally upon supports at the ends, sag down in the middle, due to their own weight; sandstone pillars under enormous weights have slowly bent without breaking. All these are examples of viscous substances.

57. Coefficient of Viscosity. Poiseuille's Law. Internal friction likewise determines the velocity with which a liquid under pressure will flow through a tube of narrow bore. Since the layer of liquid in contact with the tube remains practically at rest the velocity of flow increases from without inward, becoming a maximum at the center. With a definite difference of pressure between the two ends of the tube, the liquid assumes a steady state of flow and the moving mass may be considered as made

up of a number of concentric liquid tubes slipping over each other under pressure and retarded by molecular friction. With a difference of pressure p, between the two ends of the tube, the quantity Q, discharged through a tube of length l, and radius r, in time t, is given by the equation

$$Q = \frac{\pi p r^4}{8 l \eta} \cdot t \qquad (134)$$

where η is called the *coefficient of viscosity*.

This relation, first determined experimentally by Poiseuille, a French physician, and subsequently derived theoretically by others is known as *Poiseuille's law*. It has been found to hold very accurately for small velocities of flow, but fails when the velocity exceeds a certain value depending upon the size of the tube and the nature of the liquid. Poiseuille's law is of importance in questions relating to the capillary circulation of the blood.

The coefficient of viscosity η, is defined as the tangential force exerted upon unit area of either of two horizontal planes placed at unit distance apart, in the viscous substance, one of the planes being fixed and the other moving with unit velocity. This coefficient may be measured in several other ways, by determining the resistance encountered by a solid in moving through the viscous liquid.

The value of η varies greatly for different substances. For water at 15° C., its value is 0.0115; for glycerine it is about 300 times as much, for ether one third as much and for air one fifty-fifth as much. The viscosity of liquids usually diminishes with increase of temperature. Hot water is less viscous than cold; syrups and saline solutions are limpid when hot and viscous when cold. In gases on the other hand, the viscosity *increases* as the temperature increases.

CHAPTER VII

MECHANICS OF FLUIDS

(A) FLUIDS AT REST

58. Fluid Pressure. A perfect fluid is a body which would offer no resistance to a shearing stress. This, as we have seen, is a purely ideal case but if we consider the distortions in a fluid mass as vanishingly small, even the slight resistance of the fluid to change of form, arising from viscosity becomes zero, and we find that *fluids at rest behave as perfect fluids.* Since in a perfect fluid there is no viscosity, and hence no statical friction, it follows that the *pressure* on the containing vessel must all be *normal* to the *surface* of the vessel, because any pressure other than a normal pressure could be resolved into a normal and a tangential component. The normal component would produce pressure on the side of the vessel; the tangential component would produce

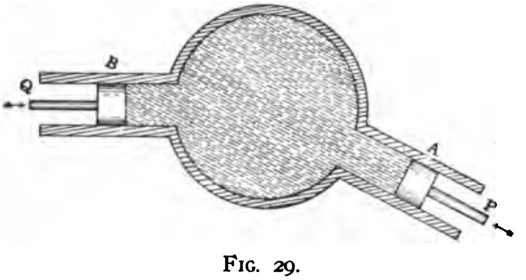


FIG. 29.

motion in the fluid unless opposed by statical friction. But there is no statical friction, and since the fluid *is at rest,* there can be *no tangential component.* Therefore the fluid pressure on the side of a vessel is normal to the side of the vessel.

Again, it follows from the absence of statical friction in a fluid, that any pressure exerted upon any surface of a confined fluid must be transmitted undiminished to every equal area of

surface in contact with the fluid. For consider two cylinders connected with a closed vessel, (Fig. 29), each cylinder being stopped by a piston of area *a*, and the whole filled with a fluid. If now one piston be forced in through a distance *b*, then the second piston must be driven out an equal distance, if no compression of the fluid is to ensue. The work done by the first piston is $P\,b$, where P is the total force exerted. Since no work is done in compressing the liquid, the work done on the second piston must be $Q\,b$.

Whence

$$P\,b = Q\,b \qquad (135)$$

or

$$P = Q \qquad (136)$$

59. Pressure at any Point in a Fluid. Since pressure in a fluid is transmitted undiminished in all directions it follows that the pressure at any point is the same in all directions. For suppose a body of liquid to be in equilibrium, and conceive a cylinder of the fluid of very small diameter to become solid without change of density. Then unless the pressures upon the ends of the cylinder are equal and oppositely directed, motion will ensue. But the liquid is at rest; hence there is no difference in the pressures upon the two ends. Now since the cylinder may be rotated in a horizontal plane, it follows that at any point in a liquid the horizontal pressure is the same in all directions, and since the diameter of the cylinder may be indefinitely diminished, the vertical pressures must be equal for the same reason.

This may be illustrated experimentally by the apparatus shown in Fig. 30. Bend three glass tubes of about 5 mm. diameter, into the forms of tubes *I, II, III*. Fill the lower parts of the three tubes to the same hight with mercury and immerse in a vessel of water, keeping the mouths of the three tubes in a horizontal line. The difference in level of the mercury in the two arms of each tube is the same; thus showing that in that horizontal plane in the liquid, the upward, downward and lateral pressures are equal. On lowering the tubes to

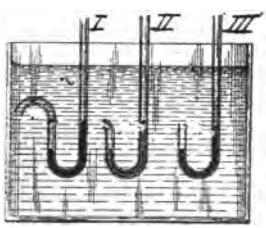


FIG. 30.

the bottom this difference in level increases; thus showing that the pressure increases as the depth increases.

60. Free Surface of a Liquid Horizontal. It may readily be shown that the free surface of a liquid at rest is horizontal. Let A (Fig 31), be a particle of liquid of mass m, on a surface that is *not* horizontal. Then the force of gravity on the particle is $m\,g$ in the direction $A\,B$. This can be resolved into a normal component $A\,C$, and a tangential component $A\,D$. The normal component simply produces pressure on the interior of the liquid and may be neglected. The tangential component $A\,D$ is free to produce motion. Its value is

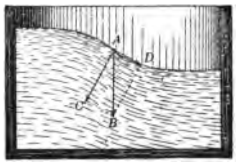


FIG. 31.

$$AD = Mg \cos BAD \qquad (137)$$

But by hypothesis the liquid is at rest; this can not be true unless $AD = 0$. But since $M\,g$ cannot be zero, $\cos BAD$ must be zero; that is, $BAD = 90°$, or the surface of a liquid at rest must be horizontal.

61. Pressure of a Heavy Liquid on an Immersed Surface. Consider a surface of area A, immersed in a liquid of density d. It is required to find the pressure upon this surface due to the weight of the liquid. Let the area be subdivided into a large number of very small areas, $a_1\ a_2\ a_3 \ldots\ldots a_n$, then

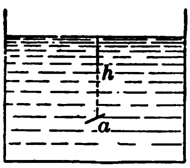


FIG. 32.

$$A = \Sigma\, a \qquad (138)$$

If we take one such little area as a. (Fig. 32), at a depth h from the surface, then the mass of liquid pressing upon this surface is the volume $a\,h$, multiplied by the density, or $a\,h\,d$, and the *force* or *pressure* on this small surface is $a\,h\,d\,g$. The total pressure on the surface is therefore

$$P = a_1h_1dg + a_2h_2dg + a_3h_3dg + \ldots\ldots a_nh_ndg$$
$$= (a_1h_1 + a_2h_2 + a_3h_3 + \ldots\ldots\ldots a_nh_n)\,dg \qquad (139)$$

Let H be the distance from the center of gravity of the

surface pressed upon, to the surface of the liquid. Then by equation (82) page 41, we have

$$H = \frac{\Sigma a h}{\Sigma a} \tag{140}$$

or

$$P = A H d g \tag{141}$$

This is a general expression for the pressure due to a heavy liquid upon any surface immersed in it, in any position whatever. The expression shows also that the pressure upon an immersed surface varies directly as the area of the surface, as the depth of the center of gravity below the surface of the liquid, as the density of the liquid, and as the acceleration due to gravity.

62. Principle of Archimedes. *A solid immersed in a liquid is buoyed up by a force equal to the weight of the liquid displaced.* This principle is readily shown to be true by considering a vessel filled with liquid which is in equilibrium. This condition of affairs will not be disturbed if we suppose a certain part of the liquid to become solid without change of density. The forces acting on the solidified portion are its weight acting through its center of gravity, and the pressure of the liquid upon its outer surface. Since this equilibrates the weight of the solidified portion, it follows that the resultant of all the pressures due to the fluid, must be directed upward through the center of gravity of the solidified portion, and equal to the weight of that portion. Consequently, the buoyant force exerted upon any body immersed in a fluid, is equal to the weight of the fluid displaced by the body, and acts through the center of gravity of the body.

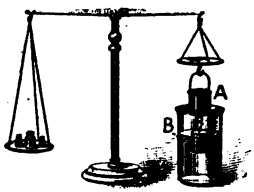


FIG. 33.

A result of this fact is that any body weighed in a fluid, weighs less than if weighed in air, by an amount equal to the weight of the fluid displaced. This seeming loss of weight is due to the buoyant force of the fluid in which the body is immersed. Hence to find the volume of any body of irregular outline it is only necessary to weigh it first in air and then in distilled water at 4° C. The apparent loss of weight is the weight of a volume of water, equal to the volume of the body, and if

the weight be expressed in grams, the volume is at once obtained in cubic centimeters. In this way the number of cubic centimeters of iron in a handful of carpet tacks may be obtained.

In accordance with the principle of Archimedes, it is necessary to correct for the buoyancy of the air, in refined determinations of mass. This precaution is unnecessary in case the object weighed and the weights used have the same density; it is the more necessary the smaller the density of the body weighed.

If we immerse a body in a fluid whose density is greater than that of the body, the body will displace more than its own weight of the fluid and will therefore rise until the weight, of the fluid displaced just equals its own weight, when it floats. Hence a floating body displaces its own weight of the liquid in which it floats.

Finally, if we call m the mass of the body,

m_1 its apparent mass when weighed in the fluid,
d the density of the body,
d_1 the density of the fluid,
v the volume of the body, and if we set the apparent loss

of weight equal to the buoyancy of the fluid we have

$$m\,g - m_1 g = v\,d_1 g \tag{142}$$

63. Density and Specific Gravity. *The density of a body is its mass per unit volume.* The specific gravity of a body is the ratio between the density of that body and the density of some substance assumed as a standard. For liquids and solids the standard is distilled water at a temperature of 4°C. For gases hydrogen is usually assumed as the standard. If the mass and volume of a body be known its density is given by the relation

$$d = \frac{m}{v}$$

Ordinarily the volume of a body is found in accordance with the principle of Archimedes, by finding the mass m, of the body in air and its apparent mass m_1, in some fluid of known density, d_1. The difference is the mass in grams of the fluid displaced by the body. The density of the body will then be given by the

expression obtained in the previous article, on substituting for v, its value m/d, or

$$d = \frac{m}{m - m_1} \cdot d_1 \tag{143}$$

In case the fluid is distilled water at 4°C., d_1 is taken as unity. If the weighing be done at some temperature other than 4°C., or the fluid be other than distilled water, then d_1 represents the density of the fluid used, *at the temperature at which the weighing was done.*

For bodies lighter than water, a sinker with a sharp point is hung from the left hand pan, immersed in the standard fluid and counterpoised by small shot or fine sand. The body whose density is sought is placed in the left hand pan and its mass determined as usual. It is then fastened to the sinker by sticking it upon the sharp point, immersed in the fluid and equilibrium restored, either by adding masses to the left hand pan, or removing them from the right hand pan. The change in mass represents the mass of the liquid displaced, i. e., $(m - m_1)$. Whence as before

$$d = \frac{m}{m - m_1} \cdot d_1$$

For finding the density of liquids, the most accurate method is by means of the pyknometer, or "specific gravity bottle." This is a small flask, fitted with a ground glass stopper, which is perforated throughout its length for the escape of superfluous fluid on filling the flask. The flask is first cleaned and dried and weighed empty. It is then filled with the liquid and weighed, and finally filled with the standard and weighed. These two weights, less the weight of the empty flask, give at once the masses of equal volumes of the substance and of the standard; whence we have

$$d = \frac{m}{v}$$

Other methods, of which one or two will be described, are frequently employed for determining the density of liquids especially when great accuracy is not required.

64. Liquids in Communicating Tubes. Consider a bent tube (Fig. 34), containing two liquids, between which there exists no chemical attraction. When the system has come to rest, it will be found that the liquid stands at a hight h, above a horizontal line through the junction of the two liquids. The weight of this column is evidently balanced by the weight of h_1, the hight of the denser liquid above the same line. Let d and d_1 be the respective densities of the two liquids, and let a be the area of cross-section of the tube. The equation of equilibrium is then

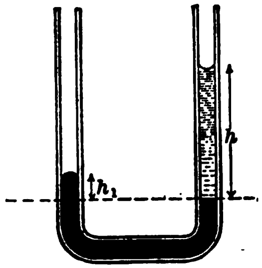


FIG. 34.

$$a\, h_1\, d_1\, g = a\, h\, d\, g \tag{144}$$

whence

$$\frac{d_1}{d} = \frac{h}{h_1} \tag{145}$$

or the hights of the two liquids above their common plane of separation, vary inversely as their densities.

In case the liquids react chemically upon each other, the device shown in Fig. 35, may be used. The bent tube is inverted and the ends placed in cups containing the liquids of densities d and d_1. Through a short tube at the top the air may be partly removed from the tubes, producing a difference of pressure P, between the air inside and outside the tube. This difference is balanced in each case by the rise of the liquid in the two tubes, and we have for equilibrium

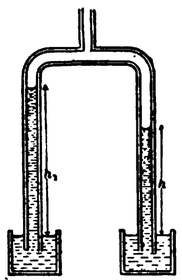


FIG. 35.

$$a\, h_1 d_1 g = a\, h\, d\, g$$

as before.

If the density of one of the liquids be known, the density of the other may be computed at once.

65. Hydrometers. An hydrometer is an instrument for determining the density of a substance quickly, and to a moderate degree of accuracy. Its action depends upon the principle that *a floating body displaces its own mass of the fluid in which it floats.* Hydrometers are of two kinds—hydrometers of variable immersion, and hydrometers of constant immersion.

(*a*) *Hydrometers of variable immersion.* The hydrometer of variable immersion consists of a hollow body, (Fig. 36), furnished with a slender uniform stem, which is usually graduated in equal divisions. The lower end of the instrument is weighted with mercury or fine shot so that it will float upright and sink to a certain fixed mark upon the stem, when immersed in a certain fluid of known density d.

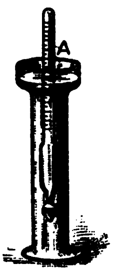


FIG. 36.

The hydrometer is calibrated empirically and is usually provided with a scale giving the densities directly in terms of the density of water. For liquids heavier than water, the fixed mark is near the top of the stem, while for liquids lighter than water it is placed near the bottom.

Hydrometers of variable immersion find wide application in manufacturing operations, where they are used to show the approximate density of the liquids in question. Such instruments are generally graduated with special reference to their use, and take the forms of lactometers, alcoholometers, acidimeters, salinimeters, etc.

(*b*) *Hydrometers of constant immersion.* The Nicholson hydrometer (Fig. 37), is an instrument of much more general application, since it may be employed to determine the densities of either solids or liquids. It consists of a hollow body weighted as before and provided with a slender stem carrying at its upper end a small pan for the reception of known masses. The stem is provided with a single mark A, to which the instrument is sunk in all cases by means of gram masses placed in the small pan above.

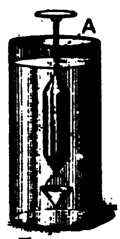


FIG. 37.

Let M be the mass of the hydrometer, and let m_1 and m_2 be the masses required to sink the instrument to the mark in liquids of densities d_1 and d_2. Then, since the volume of liquid displaced is always the same, we have

$$\frac{M + m_1}{d_1} = \frac{M + m_2}{d_2} \tag{146}$$

or

$$d_2 = \frac{M + m_2}{M + m_1} \cdot d_1 \tag{147}$$

If the first liquid be distilled water at 4°C, then

$$d_2 = \frac{M + m_2}{M + m_1} \tag{148}$$

The mass of the instrument is determined once for all and forms the constant of the instrument.

The hydrometer may also be used to determine approximately the mass of a small body, by placing it upon the pan, and noting the mass necessary to be added to sink the instrument to the mark; the body is then removed and additional masses added to sink the instrument again to the mark.

At the lower end of the instrument is provided a basket for holding a solid, when immersed in water. If the mass of the body m_b, has been determined as described above, and m_1 and m_2 be the masses to be added to sink the instrument to the mark when the body is on the pan, and in the basket respectively, then the loss of weight of the body in water is

$$(M + m_b + m_2) - (M + m_b + m_1) = m_2 - m_1 \tag{149}$$

and the density of the body is

$$d = \frac{m_b}{m_2 - m_1} \tag{150}$$

66. The Barometer. A barometer is an instrument for measuring the pressure of the atmosphere. It is made by taking a stout glass tube (Fig. 38), about 80 cm. long and closed at one end, filling it with mercury, and inverting it with the open end under the surface of mercury in a shallow dish. When the system has come to rest, the mercury in the tube stands 76 cm. above the level of the mercury in the dish, if the experiment is performed at sea level.



FIG. 38.

Now since pressure in fluids is transmitted undiminished in all directions, it follows that the weight of this column of mercury, 76 cm. in hight, must be balanced by the downward pressure per unit area, of the air upon the surface of the mercury in the dish. If for any reason the atmospheric pressure change, the corresponding difference in the hight of the mercury in the tube enables us to measure this change. If the barometer be carried up the side of a mountain, or down into a mine, the elevation above or the depression below

the sea level may be roughly determined from the difference in the barometric readings.

The pressure of the atmosphere as measured by the barometer is readily computed. Assume the tube (Fig. 38), to be of unit cross-section. Then the volume of mercury supported is 76 cubic centimeters; the mass is $76 \times 13.596 = 1033.296$ grams, and the force is $1033.296 \times 980 = 1{,}012{,}630$ dynes per square centimeter. This pressure is called the *pressure of one atmosphere.* In English units, the pressure of one atmosphere, is approximately equal to a weight of 14.7 lbs. per square inch.

It should be noted that the barometric hight is constantly changing, and that the standard hight of 76 cm. or 30 inches is for places at the *level of the sea and at the temperature of 0°C.* An increase in the elevation above sea level produces a rapid change in the barometric hight. Ann Arbor is 882 feet above sea level, and the mean barometric reading for the year 1901 was 29.03 in.

67. Manometers. Let a tube be attached to a bell jar and the jar be placed upon the plate of an air pump with the lower end of the tube dipping into a cup of mercury, as shown in Fig. 39. On withdrawing the air from the bell jar the mercury rises in the tube, thus furnishing a measure of the exhaustion as the pumping proceeds. Such an arrangement is called *a manometer.* For measuring slight differences in gaseous pressure, the second form shown in Fig. 39, is used, where the total pressure upon the gas in the horizontal arm is one atmosphere, plus the mercury column AB.

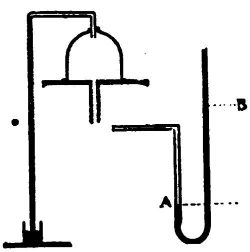
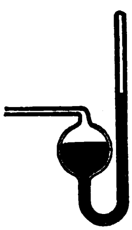


FIG. 39.

For the measurement of pressures amounting to several atmospheres the *compression manometer* (Fig. 40), is a convenient and compact form.


FIG. 40.

68. Pumps. In the action of the ordinary lifting pump (Fig. 41), we have an application of the pressure of the atmosphere. This pump consists of a piston D, working air tight in a cylinder HK, to which

is attached a pipe of smaller diameter, closed by the valve *A*. Suppose the pump to be inserted in the water at *L*, and the piston to be at the bottom of the cylinder. On raising the piston the weight of the air closes the valves *B* and *C* in the piston, and the air in the small pipe expands, raises the valve *A*, and passes in to fill the vacant space in the cylinder. The pressure of the air in the pipe is diminished and the hydrostatic pressure due to the weight of the atmosphere drives the water up into the small pipe. On the downward stroke the air in the cylinder closes the valve *A*, and escapes through the valves *B* and *C* into the outer air. By the second and succeeding strokes of the piston, the air below is still further rarefied, until the water rises above the valve *A*, passes above the piston through the valves *B* and *C* on the next downward stroke, and is finally lifted to *S* where it flows out. Since the density of mercury is 13.6 times that of water, it follows that the pressure of the atmosphere will support a column of water 13.6 times the barometric hight; i. e., 30 in. $\times$ 13.6 = 34 feet. This would represent the maximum effect to be obtained from the pressure of the air; as a matter of fact an ordinary pump will not raise water much more than 26 feet.

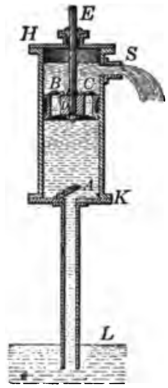


FIG. 41.

The *force pump* is provided with a solid piston and a second pipe is attached to the cylinder above the valve *A*. This pipe leads to an air chamber into which the water is driven on the down stroke of the piston, and retained there by a valve similar to *A*. An outlet pipe fitted with a nozzle delivers the water driven out of the air chamber by the force of the compressed air; in other respects the working of the pump is similar to that of the lifting pump.

69. The Siphon. The Siphon (Fig. 42), is an instrument for transferring liquids from one vessel to another at a lower level. It consists of a bent tube, with arms of unequal length filled with liquid and inverted with the shorter arm in the vessel from which the liquid is to be transferred. The distances *a* and *b*, from the surfaces of the liquid in the vessels to the highest point of the bend of the tube represent the two columns of liquid in motion.

Let h be the hight of the column of liquid which the atmospheric pressure will support at time and place of the experiment, let s be the cross-section of the tube and d the density of the liquid. Then the pressure to the right, on a plane through the highest point in the tube is

$$hsdg - asdg \quad (151)$$

and the pressure to the left is

$$hsdg - bsdg \quad (152)$$

or the difference in pressure, tending to produce flow is

$$(hsdg - asdg) - (hsdg - bsdg) = (b - a)\,sdg \quad (153)$$

From this it appears that the velocity of flow will be proportional to the difference in the length of the arms. If a equal b the flow ceases. If a be greater than b, the liquid flows in the opposite direction.

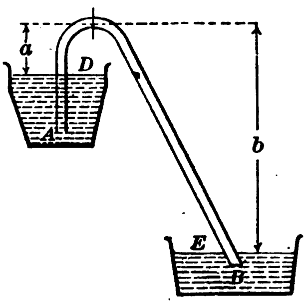


FIG. 42.

70. The Air Pump. The action of the air pump is essentially the same as that of the common lifting pump.

In addition to the parts described in Article 68, the air pump is fitted with an automatic valve A (Fig. 43), which opens at the up stroke of the piston and closes on each down stroke. This is effected by making the valve A, in the form of a conical plug attached to a rod passing through the piston, which slides upon the rod with a small degree of friction. By this means the valve A is opened to admit air from the receiver, which by reason of its diminished density would be unable to lift the valve by its expansive force. A manometer F, shows the degree of exhaustion in the receiver as the action of the pump proceeds. Owing to unavoidable wear and consequent leakage in the pump itself it is impossible to secure a high degree of exhaustion with a mechanical air pump of the ordinary type.

For work requiring a high degree of exhaustion as in the production of Geissler, or Roentgen tubes, a mercury air pump

is employed. Such pumps are usually made almost entirely of glass, and the piston is a column of mercury, which on rising,

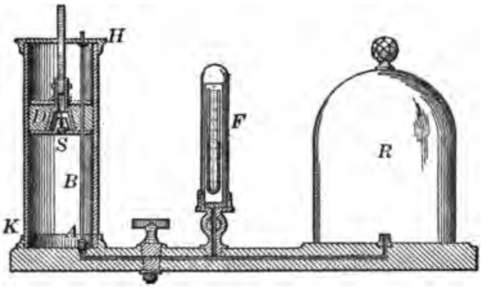


FIG. 43.

fills a large bulb expelling the air through a barometric manometer. On descending the air from the vessel to be exhausted enters the bulb through a glass-mercury valve, and expanding fills it at a much diminished pressure, only to be forced out as before by the liquid piston. By the use of such pumps pressures of one millionth of an atmosphere are readily attainable.



FIG. 44.

71. The Geryck Air Pump. While mechanical pumps are desirable on account of their rapid action, yet even at best they can not be depended upon to produce a vacuum below two or three millimeters of mercury. On the other hand the mercury pump is extremely fragile and so slow in action as to be practically useless in many important problems. In the Geryck air pump (Fig. 44), we have a combination of the good features of both forms, and consequently a most rapid and efficient pump.

A longitudinal section through one barrel of the pump (Fig. 45), shows the inlet tube at *A*, leading to the receiver to be exhausted. At *B* the air is admitted to the cylinder above the piston. The cylindrical bucket leather *C*, attached to the piston is held firmly against the walls of the cylinder by the pressure of the oil *J*, in the annular space *D*. The ordinary piston valve *E*, permits the air to rise above the piston on the down stroke as in the older forms of pump, but is useful only at the beginning of the exhaustion, remaining seated after the vacuum reaches 15 mm. of mercury. The auxiliary air pipe *F*, leading into chamber *B*, is likewise to relieve the piston at the first few down strokes.

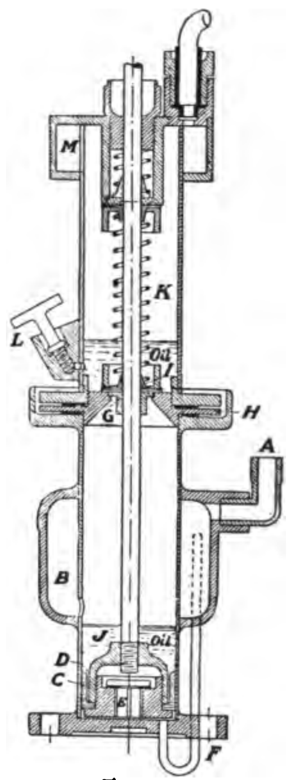


Fig. 45.

The upper valve is represented by the collar *G*, which fits the piston rod snugly and is closed above by the hydraulic collar *I*, and a layer of oil several centimeters in depth. The collar *G*, is held firmly pressed into its seat by the spiral spring encircling the piston rod.

When the piston is at the bottom of its stroke the air from the receiver enters the cylinder freely at *B*, but as the piston rises the port *B* is closed and the entrapped air is carried upward through the valve *G*, by the liquid piston of oil *J*, which at the highest point of the stroke escapes into the cylinder *K*, carrying all air with it. The valve *G*, is automatically lifted some six millimeters from its seat by the head of the piston at each up stroke, thus forcing the air and oil into *K*. On the down stroke six millimeters of oil are allowed to reënter the lower cylinder before the valve *G* closes. Should any oil get past the cylinder bucket *C*, it is picked up through the valve *E*, at the next stroke. *L* is a plug for introducing oil into

the cylinder *K*. The oil should fill *K* up to the level of the overflow.

Through the outlet tube at the top the air either escapes into the outer air, as in a single pump, or is drawn out by a second pump which exhausts from the cylinder *K*, as in the Duplex pump shown in Fig. 44. With the two pumps thus working in series a vacuum of 0.0002 mm. may be obtained. The action of the pump is rapid and the vacuum is held for any length of time. It is important that a good drying tube filled with P_2O_5, be at all times attached to the inlet pipe *A*.

72. Weight and Density of Air. Galileo satisfied himself that air had weight by weighing a glass globe first filled with air at ordinary pressure, and then filled with air under high pressure. The experiment is ordinarily performed today by weighing a stout glass balloon, when filled with air and then when the air has been exhausted. A liter of ordinary air, containing 0.04% of carbon dioxide, under standard conditions, i. e., at 0°C. and 76 cm. pressure, weighs 1.293 grams. Hence the density of air is 0.001293 grams per cubic centimeter.

In the use of a delicate balance it is frequently necessary to correct the weighings for the buoyancy of the air in which the substance is weighed, in accordance with the principle of Archimedes. Thus let w be the weight of the body weighed, w' the weight of the brass weights, and a and a' the buoyant force of the air upon the body and weights respectively. Then for equilibrium

$$w - a = w' - a'$$

or

$$w = w' + a - a' \qquad (154)$$

Thus the correction to be added to the weight of 25 grams of cork, whose density is 0.25 g. per cc. is obtained as follows:

$$a = \frac{25}{0.25} \times 0.001293 = 0.1293 \text{ grams}$$

$$a' = \frac{25}{8.5} \times 0.001293 = 0.0038 \text{ grams}$$

Or the correction to be added is 125 milligrams.

73. Boyle's Law. Mention has already been made of the work necessary to compress a gas in a cylinder, and of the resist-

ance offered by a gas to any force tending to *decrease* its volume. The relation between the volume of any mass of gas and the pressure exerted by the gas upon the walls of the containing vessel was first stated by *Robert Boyle* in 1662, although its far reaching importance was more fully realized by *Mariotte*, a French physicist, who rediscovered the law, independently, in 1676.

Boyle's law states that, *at a constant temperature, the volume of a body of gas varies inversely as the pressure to which it is subjected.* Thus if p_0 and v_0 be the initial pressure and volume for a mass of gas, and p and v, the final pressure and volume, then

$$p\,v = p_0 v_0 \qquad (155)$$

or *for a constant temperature, the product of the pressure and the volume is a constant.* Since the density of any body varies inversely as the volume, the law may be stated in another way, viz:

$$\frac{p}{d} = \frac{p_0}{d_0} \qquad (156)$$

or *the pressures are directly proportional to the densities.*

Careful experiments have shown that for high pressures or low temperatures, Boyle's law is only approximately true. Thus it has been shown that all gases may be liquefied by the application of pressure and the reduction of the temperature. As the gas approaches the point of liquefaction, the variation from the law is most noticeable. Gases which can be liquefied by pressure at ordinary temperatures, such as chlorine, sulphur dioxide, and carbon dioxide can hardly be said to obey the law at all, for pressure exceeding 100 atmospheres; while for gases like nitrogen, oxygen and hydrogen, the law is very nearly true at ordinary temperatures.

An ideal gas would obey Boyle's law at all temperatures, and in general, it may be said, that the farther removed any gas is from its point of liquefaction, the more nearly it behaves as an ideal gas. This departure from the law by all gases when near the point of liquefaction, has been attributed to the action of two causes; (*a*) *The attraction of the molecules of the gas, for each other.* This would increase as the compression increases and

therefore assist the compression. (*b*) *The size of the molecules*, which would tend to retard the compression of the gas. More complete formulae for the behavior of a gas under varying pressure and temperature have been proposed by Van der Waals, Clausius and others.

(*B*) FLUIDS IN MOTION.

74. Velocity of Efflux. When a small opening is made in the side of a vessel filled with liquid, at a distance h below the surface, the liquid flows out with a definite velocity v, where

$$v^2 = 2\,g\,h$$

This formula for the velocity of a liquid issuing under pressure due to a head h, is known as *Torricelli's theorem.* It may be demonstrated as follows: Suppose a particle of liquid of mass m, to be situated in the surface of the liquid. Its potential energy with respect to the orifice is $m\,g\,h$. In passing from the surface of the liquid to the orifice it has fallen a distance h, and, if we neglect the viscosity of the liquid, its kinetic energy on emerging from the orifice, must equal its potential energy at the beginning, or

$$\frac{m\,v^2}{2} = m\,g\,h \qquad (158)$$

whence as before,

$$v^2 = 2\,g\,h$$

If a be the area of the orifice, then Q, the quantity of liquid discharged in time t, would be

$$Q = a\,v\,t \qquad (159)$$

In practice this rate of discharge is never reached. If the opening be a simple orifice in the side of the vessel, without mouth-piece of any sort, the quantity of liquid discharged is about 0.62 of the theoretical value. This difference is due to the convergence of the lines of flow, producing a contraction of the jet just outside the orifice, whereby the actual cross section of

the stream at this point is much reduced. By using a short cylindrical mouth-piece, of length two or three times its diameter, and set flush with the inside of the vessel, the flow may be made 0.82 of the theoretical value; while by so shaping the mouth-piece as to conform most closely to the form of the contracted jet, a velocity but little short of $\sqrt{2gh}$ is attained.

75. Velocity of Effusion for Gases. Consider a vessel filled with gas, and having an orifice at one side of cross section a, from which the gas escapes with a velocity v. Let the density of the gas be d grams/cm.^{3}, and the pressure be p dynes/cm.^{2}, above that of the air. The volume of gas delivered in t seconds will then be avt cubic centimeters, and the mass $avtd$ grams. The kinetic energy of the escaping gas will be $\frac{av^3td}{2}$ ergs. If the gas be allowed to do work by pushing back the surrounding air, then the work done on the air must equal the kinetic energy of the gas, if we assume the outflow to take place so slowly that the gas is not cooled below the temperature of the surrounding air. The work will then be the product of the force ap, and the distance vt, or

$$\frac{av^3td}{2} = avtp \qquad (160)$$

whence

$$v = \sqrt{\frac{2p}{d}} \qquad (161)$$

If the pressure p, be expressed in terms of a column of water h centimeters high, then the formula becomes

$$v = \sqrt{\frac{2gh}{d}} \qquad (162)$$

or the velocity with which a gas effuses through an opening varies directly as the square root of the difference of pressure on the two sides of the opening and inversely as the square root of the density of the gas.

From this it follows that two gases effuse through the same

opening, under the same difference of pressure, with velocities inversely as the square roots of their respective densities. Bunsen has utilized this principle to compare the densities of gases by observing the time required for the same volume of the various gases to effuse through the same opening under the same difference of pressure.

76. Flow of Liquids through Tubes. If instead of a metal tube, an elastic tube of rubber be attached to the orifice of the discharging vessel, the efflux is the same as that by a rigid tube of the diameter assumed by the elastic tube, *so long as the flow is uniform.* If, however, the flow by any means be made to assume an intermittent or pulsating character, the discharge from the elastic tube is notably larger than from the rigid tube. The nature of the discharge is also modified, in that the stream from the rigid tube reproduces faithfully every feature of the pulsating impulse, while the elastic tube rapidly smooths out the inequalities of pressure, so that in an elastic tube of sufficient length the pulsation disappears entirely. This fact is of importance in explanation of the flow of the blood through the arteries, the coats of which are extremely elastic.

Again the flow of liquids through tubes is much retarded on account of friction, not only among the particles of the liquid but between the liquid and the walls of the tube. This latter friction is much the more important of the two, and increases rapidly with the roughness of the walls of the tube. The flow of a river is greatest at the center of the stream, and at the surface of the water where the effect of friction from the bed and banks is as small as possible.

In a vertical tube the liquid column breaks into a series of liquid masses fitting the tube more or less perfectly. These masses acquire an increasing velocity in their descent and act as liquid pistons fitting the interior of the tube. A partial vacuum results and the water is forced into the pipe more rapidly on this account. The effect of this exhaustion causes the noisy draught with which the last portions of water leave a wash basin or a bath tub, where the waste pipe is long and free. This action of vertical waste pipes is also liable to draw out the water from

the siphon traps, and leave the way open for the entrance of poisonous sewer gas. For this reason all waste pipes from basins, closets, bath tubs, etc., are now required to have separate vent to the outside air.

This reduction of pressure by liquids in vertical tubes is utilized in the Bunsen air pump where a vertical column of water of more than 34 feet is made to exhaust the air from a receiver; the limit of the exhaustion being of course the pressure equal to the vapor tension of water at the existing temperature. In the Sprengel pump the liquid is mercury. This has two advantages: it requires a vertical column but 30 inches long, and the vapor tension of mercury is practically negligible.

77. Flow in Pipes of Variable Section. In a tube of variable cross section, the flow of liquids presents some interesting features. In Fig. 46 the variation of the pressure exerted by the fluid upon the walls of the tube is shown by the manometer tubes. It is thus seen that in a tube of variable cross-section running full of liquid, the pressure is greatest in the widest part of the tube and least in the narrowest part. This somewhat surprising result is easily explained by considering the conditions of flow in the different parts of the tube. It is readily seen that with steady flow the velocity is greatest where the cross-section of the stream is least, and *vice versa*. The liquid in passing from a wider to a narrower part of the tube must, therefore, undergo an acceleration, since for steady flow, the same volume must pass any section in the pipe in the same time. To produce this acceleration the pressure on any section must be greater from *behind*; similarly in passing from the narrower to the wider part of the tube the velocity *decreases*, or the acceleration is *negative*; hence the pressure is greater from *before* than from *behind*, so that the pressure is greatest in the widest part of the tube and least in the narrowest part.

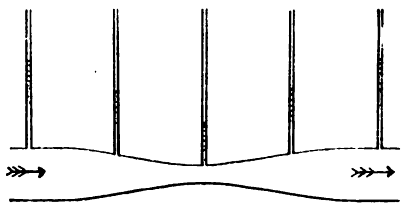


FIG. 46.

78. Jet Pump. The reduction of pressure within a contracted stream has been applied in the construction of many useful pieces of apparatus. In Fig. 47 is seen a common form of aspirator or jet pump as used in the laboratory. Water entering through the arm *B*, under hydrant pressure passes through the constricted inner tube at *A*, and flows out at *D*. Owing to the small cross-section of the stream at *A*, the velocity is very great, and the pressure is so much reduced that the air from the tube *E* is drawn along through the constricted portion in a torrent of small bubbles and carried down the tube *D*. With a well constructed pump of this kind, a vacuum of from 25 to 27 inches of mercury may be obtained, with water from the city water mains. Obviously the pump will work equally well if the water enters at *E* and the air through *B*.

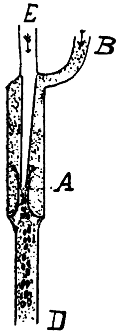


FIG. 47.

This pump has been adapted to numerous uses. The filter pump, of the laboratory, the atomizer for spraying of perfumes or medicines, the injector in steam boilers, and the forced draught as used on locomotives are all different forms of this apparatus.

PROBLEMS

1. Find the total pressure on the bottom of a tank 10 feet square and five feet deep, when full of water. Find the pressure on one side of the same tank.

2. A triangular plate is immersed vertically in the water, with the vertex in the surface of the water and the base horizontal. The hight and base of the triangle are each 50 cm.; find the pressure on the face of the plate.

3. What is the density of a body whose mass is 678 gms., if it weigh 235 gms. when immersed in a fluid whose density is 1.94 gms. per cc.?

4. A piece of wood of density 0.6 gm. per cc., floats on water. The volume of the wood is 40 cubic centimeters. What is the volume of the water displaced?

5. A body having a density of 2.35 gms. per cc., weighs

624 gms. when immersed in a liquid whose density is 0.827 gm. per cc. What is the mass of the body?

6. If the density of ice be 0.9179 gm. per cc., and of sea water be 1.025 gm. per cc., what portion of an iceberg is above water?

7. A piece of silver and a piece of gold ore are suspended from the ends of a balance beam having equal arms. The balance is in equilibrium when the silver is immersed in alcohol (sp. gr. 0.85), and the gold in ntric acid (sp. gr. 1.5); if the specific gravities of the gold and silver be 19.3 and 10.5 respectively, find the ratio of their masses.

8. A U-tube is partly filled with water. How many centimeters of oil having a density of 0.79 gm. per cc. must be added in order to raise the water in one leg 4.5 cms. above its first level?

9. Twenty-four cc. of gas at a pressure of 71 cm. of mercury, would have what volume under a pressure of 76 centimeters?

10. A liter of air under normal conditions of temperature and pressure weighs 1.293 grams. Find the weight of the air in a liter flask when the barometer stands at 72 cm., the temperature being 0°C.

11. To what depth must a diving bell 6 feet high be immersed in order that the water may rise 4 feet within it?

12. A glass tube used for sounding is 38.1 cm. long and open at the lower end. The inside is covered with a soluble pigment, and the tube lowered to the bottom, in sea water, density 1.03 gm. per cc. On raising it to the surface it is found that the water had entered the tube to a depth of 23.6 cm.; find the depth of the sea water.

13. A vessel filled with water has a circular orifice 6 cm. in diameter, 298 cm. vertically below the surface of the liquid. If the water be maintained at its initial depth, by supply from without, calculate the theoretical discharge per minute.

14. A picture of mass 4 kilos, is suspended in the ordinary way by a cord passing through two hooks and over a smooth nail. The hooks are 45 cm. apart and the cord is 120 cm. long. Find the tension in the cord.

CHAPTER VIII

MOLECULAR MECHANICS

79. Molecular Forces. By molecular forces are meant all those forces whose range of action is confined to insensible distances, that is, to distances comparable to the spaces between the individual molecules of a solid or a liquid. Under this head belong the forces of adhesion, cohesion, friction, viscosity, elasticity, capillarity and surface tension, and although certain of these have been mentioned in previous topics it seems proper to classify them here under the general term *molecular forces.*

The magnitude and importance of these forces are apt to be under estimated. It is owing to the action of molecular forces that any solid body is not only kept from falling to pieces of its own weight, but is able to resist the application of enormous stress as well. A clean glass tube cautiously lowered to the surface of clean water exhibits no attraction for the water, and causes no change in its upper surface so long as there exists any appreciable distance between them. But if the glass tube touch the surface, the water promptly runs up into the tube and clings to the outside, so that when the tube is withdrawn, a drop of water hangs to the lower end and the force of gravitation is unable to pull it off. Clearly we have here to do with forces, in comparison with which the force of gravitation is weak and insignificant.

The attraction between *unlike molecules,* as between water and glass, is called *adhesion*; that between *like molecules,* as between water and water is called *cohesion.* These are in reality only different names for the same thing, viz: molecular attraction, and it is to be noted that this attraction is exhibited only so long as the substances are *in contact,* that is, it acts through insensible or molecular distances. From a study of the behavior of gases we are led to believe that elasticity is due to molecular forces and molecular motions, and that the same is equally true of capillarity, surface tension, and viscosity.

80. Adhesion and Cohesion. If two smooth, freshly cut surfaces of lead be firmly pressed together with a slight twisting motion, they cling together with considerable force, but having once been pulled apart they can be made to stick again only by application of sufficient force to bring the surfaces into close contact. A pair of glass plates, if highly polished, plane, and free from dust, may be pressed together so firmly that it is impossible to separate them without rupture. That this is not due to the pressure of the air, is shown by the fact that the plates cling together more firmly in a vacuum than in the open air, owing to the removal of the air film between the plates.

The adhesive action of glue, cement, mucilage, and such substances renders it possible to unite two bodies so firmly that they break before separating. Dissimilar substances are united with difficulty owing to their different rates of expansion or contraction when heated or cooled. Thus it is impossible to seal an iron or copper wire into a glass tube, since the metal and the glass have different rates of expansion; platinum on the other hand and certain alloys of nickel and iron may be sealed into glass, since they expand and contract at the same rate as the glass. For the same reason different kinds of glass frequently can not be made to hold when sealed together.

Gases adhere to solids with great tenacity. It is almost impossible to free a glass tube from the adhering film of air, and consequently in the making of barometers, thermometers, and vacuum tubes of any description the air film is removed from the inner surface of the glass only by repeated heating and pumping.

The cohesive force of water is illustrated by hanging a clean, smooth, glass plate to one arm of a balance so that it is horizontal, and bringing under it a jar of clean water. On touching the under side of the plate to the water, taking care to avoid air bubbles, it will be found necessary to add a relatively large weight to the opposite scale pan in order to pull the disk squarely away from the water. When the disk is pulled away it is found that the *under surface is wet,* thus showing that the attraction between the solid and the liquid is greater than that between the particles of the liquid. This is true in all cases where *a liquid wets a solid.*

If mercury be used instead of water, it is found that *a greater force* will be needed to pull the plate away from the mercury and

also that the *under side of the plate is now dry.* In the latter case we find that the attraction between the solid and the liquid is less than the attraction between the particles of the liquid, as is always the case *where a liquid does not wet a solid.*

81. Capillary Phenomena. If a solid be immersed in a liquid which wets it, the liquid rises about the sides of the solid, and the surface of the liquid is concave upward. If the solid be in the form of a tube the liquid rises into the tube to a certain hight, which varies inversely as the diameter of the tube, and forms at its upper surface in the tube, a meniscus of liquid with its concave side upward. If the liquid do not wet the solid, it is depressed about the solid, or in case of a tube, it is depressed to a certain depth, varying inversely as the diameter of the tube, below the level of the liquid in the vessel, and the surface of the meniscus is *convex upward.* Since phenomena of this class are most clearly shown in the case of fine, hair-like tubes, they are called *capillary phenomena,* from *capillus,* a hair. In Fig. 48 is shown the action when a clean glass tube is immersed in water and mercury respectively.

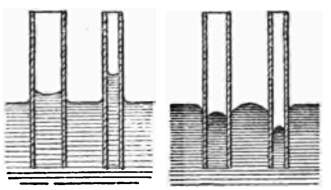


FIG. 48.

The principal facts concerning capillary phenomena are briefly these:

(*a*) In tubes of less than 2 mm. in diameter the elevation or depression varies inversely as the diameter of the tube.

(*b*) The elevation or depression is independent of the pressure to which the liquid is subjected, and also independent of the thickness of the tube.

(*c*) An *increase* in the temperature of the liquid *causes a decrease* in the elevation or depression of the capillary column.

(*d*) The elevation or depression depends upon the nature of the liquid and tube in contact. Clean water on clean glass gives an elevation greater than that of any other liquid, while pure water in a steel tube or against a silver plate gives neither elevation nor depression.

Examples of capillary action are seen in the action of blotting paper absorbing ink, in the lamp wick supplying the flame with oil, and in the swelling of a tub or barrel if filled with water when about to fall to staves. A cotton or hemp rope, if wetted, absorbs water, increases in diameter and diminishes in length; at the same time the temperature of the rope rises from 2° to 10° C. Workmen drive wedges of dry wood tightly into holes or slits cut in large stones and then wet the ends of the wedges. The increase in the size of the wedge is sufficient to burst the stone.

Besides the capillarity of liquids there seems to be an analogous phenomenon in the case of metals. Joseph Henry discovered that mercury would siphon through a bar of lead as water through a towel, and silver has been shown to pass into the pores of copper when the two metals are heated.

82. Molecular Range. In accordance with the assumption that molecular forces are exerted over insensible or molecular distances, it follows that each individual molecule becomes a center from which it exerts its molecular attractions and repulsions over the number of molecules included in its sphere of influence. Let the radius of this sphere be ϵ; then ϵ denotes the limit beyond which the molecule neither influences nor is influenced by its neighbors. Within this sphere however it is attracted equally on all sides and hence remains in equilibrium.

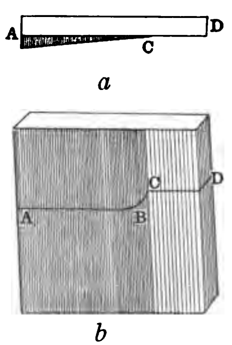


FIG. 49.

In order to determine the value of this quantity ϵ, Quincke employed a glass plate (Fig. 49 *a*), one half of which was coated with a wedge-shaped layer *AC*, of pure silver. On inserting the plate in water, with the silver film vertical (Fig. 49 *b*), the water rose against the glass above the level of the water in the dish to a definite hight indicated by *D*. At the beginning of the silver film *C*, the elevation gradually fell away with increasing thickness of the silver, until at a point *B*, where the thickness reached a value of 0.000005 cm., the angle of contact became

90° and the capillary effect disappeared entirely. At this point the molecules of the glass ceased to influence the molecules of water through the silver, hence the value of ϵ, the molecular range, is commonly given as 0.000005 cm.

83. Surface Tension. It has been shown that within the limiting distance ϵ, the molecules attract their neighbors and are attracted by them, and that a molecule situated in the body of a liquid will be in equilibrium by virtue of the equal attractions on all sides. Consider now a molecule nearer the surface of a liquid than the molecular range ϵ. In this case the horizontal attractions will be equal in all directions but the vertical attractions are unequal; the resultant being an unbalanced component toward the interior of the liquid. At the surface of the liquid this resultant stress, normal to the surface reaches a maximum, and the mass of the liquid (Fig. 50), behaves as if surrounded by an elastic bag under stress, tending to contract indefinitely and compress the liquid into as small a volume as possible.

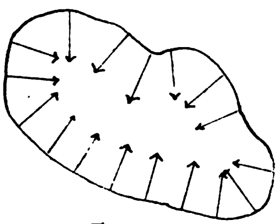


FIG. 50.

The surface of a liquid is therefore a seat of potential energy, and in order to force a molecule from the interior of the liquid out into the surface film, work must be done in moving the molecule through the distance ϵ, against the forces tending to force it back into the interior. An increase in the area of the surface film therefore, means an increase in its potential energy, and since potential energy always tends to become a minimum, it follows that if a liquid mass be freed from the action of other forces it will assume a form such that its surface will be a minimum, and its volume a maximum, that is, it will assume the form of a sphere. This condition is readily realized by placing drops of olive oil in a mixture of alcohol and water of the same density as the oil. The drops are thus freed from the action of gravity and float as spherical globules in a medium of their own density. If by any device the globule be prevented from assuming a spher-

ical form, it will still take a form presenting the minimum area consistent with the conditions imposed upon it.

The effect of surface tension is exemplified in the shape of the dew drop, in the forms of falling drops of liquid, and in the manufacture of shot, where molten lead, poured through a fine sieve at the top of a high tower, is broken up into small globules which harden as they fall through the air, and are caught in a tank of water beneath.

Again, small heavy bodies that are not wetted by a liquid, may be placed upon the surface of the liquid and float upon the surface film. Thus needles may be made to float upon water, *so long as the film is not broken,* in which case they sink at once. The same principle is illustrated in the case of small insects which run upon the surface of the water, their slight weight being insufficient to break through the surface film.

The following table shows the value of the surface tension in dynes per unit width of film, for the various substances mentioned. The values are mostly those given by Quincke.

Mercury against air	535.0
Water against air	81.0
Olive oil against air	36.9
Alcohol against air	25.5
Mercury against water	418.0
Olive oil against water	20.6
Turpentine against water	11.6

84. Experiments on Surface Tension. If two small pieces of wood be floated near each other upon the surface of clean water and a drop of oil be touched to the water between them, they fly apart to the sides of the vessel, as though drawn by a spring. The addition of the oil reduces the surface tension of the liquid film at that point, and the water film tears apart.

If a plate of clean glass be wetted with clean water, the water will spread out into a thin layer over the entire plate. If the plate be not absolutely clean, the water will gather up into irregular masses with rounded edges. If now a drop of alcohol be touched to the water layer, the water film will be seen to break at the point of contact, and gradually draw away from the

alcohol drop, leaving a dry space on the plate around that point.

If a ring of stiff wire (Fig. 51), be dipped into a soap solution and withdrawn, a film of the solution will adhere to it for several minutes. In this film, which is really *two films* placed back to back, may be seen the motion of the liquid particles seeking new positions as the tension in the film changes. If we drop a loop of silk thread upon this film, it floats about freely upon it; if the film inside the loop be broken by touching it with a hot wire, the loop suddenly flies out into a circular form, showing that the tension in the film is equal in all directions. This circular loop still floats freely in the film, and behaves like an elastic hoop of steel.

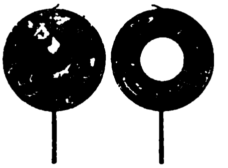


FIG. 51.

If small pieces of clean gum camphor be thrown upon the surface of clean water, the little particles begin a most lively and erratic motion. Each little piece spins with great vigor and at the same time sweeps over the surface, the larger ones gathering in the smaller ones. The gum camphor dissolves slowly in cold water, and the surface tension of the water film is thereby weakened. The spin is due to unequal dissolving on the surface of the camphor particle. If the water be warmed it spins faster; if the surface be touched with a trace of oil, the motion ceases instantly.

If several small, clean, wood balls be thrown upon clean water, they seem to attract each other and collect into a little cluster. If a number of similar balls be smoked with lampblack and then placed in the same dish, they also attract each other, but the clean balls and the smoked balls seem to avoid each other.

A small cylinder of fine wire gauze, if immersed in water and lifted out in a horizontal position, retains the water in it and may be carried about the room. On breaking the water film at one point in the gauze by blowing upon it, the water begins to run out. The flow may be checked by shaking the water so as to restore the film, thus preventing the entrance of the air.

85. Angles of Contact. From the table in Article 83, it is seen that there exists at the surface of separation between a liquid and a gas, or between two liquids, a stress or tension which

is a constant for the same substances. Thus there exists in the surface film of olive oil in contact with air, a tension of 36.9 dynes per centimeter width of the film, while for water and air the tension is 81 dynes per centimeter width. If now a liquid be brought into contact with a second liquid in the presence of air, then for equilibrium, the three surface tensions should form a triangle of forces, and theoretically the angles between the tensions should be constant. If, however, one of the forces should chance to be greater than the sum of the other two, then clearly no triangle is possible, and the system cannot come to equilibrium.

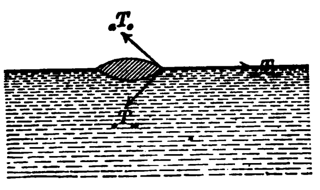


FIG. 52.

Thus in Fig. 52 is shown a drop of oil, placed upon the surface of clean water. Then at any point upon the horizontal edge of the drop there are acting the three tensions, ${}_aT_w$, between the air and the water, ${}_aT_o$, between the air and oil, and ${}_oT_w$, between the oil and water, directed in each case as indicated by the arrows. But from the table

$${}_aT_w = 81 \text{ dynes per cm. width}$$
$${}_aT_o = 36.9 \text{ dynes per cm. width}$$
$${}_oT_w = 20.6 \text{ dynes per cm. width}$$

hence

$${}_aT_w > {}_aT_o + {}_oT_w \tag{160}$$

This shows that when a drop of oil is placed upon clean water, the surface tension between air and water is so great as to overbalance the other two tensions combined, and the oil is dragged out in all directions, forming a film of infinitesimal thickness over the entire surface of the water.

If a liquid meet a solid, in the presence of air, it will in general meet it in a definite angle, which is constant for the two surfaces. This angle is called the *angle of contact,* and depends upon the nature of the substances in question. For pure water on clean glass the angle of contact is approximately zero. For pure water on clean steel or clean silver the angle of contact is about 90°. For clean mercury against clean glass it is about 132°.

86. Behavior of Films. If a film of soap solution be made to assume a curved form, there will always result a normal pressure, directed towards the concave side. It may be shown mathematically, that for a single cylindrical film, of radius R, and surface tension T, the normal pressure toward the curved side is given by the equation

$$P=\frac{T}{R} \tag{164}$$

In words this equation says, that the normal pressure in a curved film is directly proportional to the surface tension per centimeter width, T, and inversely as the radius of the film, R. In general the curvature of any surface at any point may be expressed in terms of two radii of curvature, the planes of curvature being at right angles to each other. If R_1 and R_2 be these radii, then the normal pressure of any curved film is the sum of that due to each curvature separately, or

$$P=T\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \tag{165}$$

In a soap bubble the two radii are equal, and there are also two films back to back, hence the normal pressure exerted upon the air enclosed in the bubble, is

$$P=4\frac{T}{R} \tag{166}$$

If the film is free to the air on both sides, the normal pressure must be zero. In a curved film this is possible only if

$$\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=0 \tag{167}$$

This means that

$$R_1=-R_2 \tag{168}$$

or that the radii are equal and on opposite sides of the film, that is, the film is saddle shaped.

Again since this normal pressure is directed toward the concave side and varies inversely as the radius, we are able to understand the motion of drops of water and of mercury in conical tubes. The water will move toward the smaller end of the tube,

i. e., in the direction of the greater pressure. For the same reason, the mercury globule will move toward the larger end of the tube.

If stout wire frames, representing the outlines of geometrical figures be dipped into soap solution, a large number of curious and beautiful film figures result. Whenever three such film surfaces meet along a line, the included angles will all be 120°, since the three surface tensions are all equal.

87. Capillary Action as Related to Surface Tension. Let a tube of radius r, (Fig. 53), be inserted in a liquid density d. Let the mean elevation of the liquid be h, and the angle of contact with the tube be α. Then the vertical component of the surface tension T, must be balanced by the weight of the liquid column of hight h.

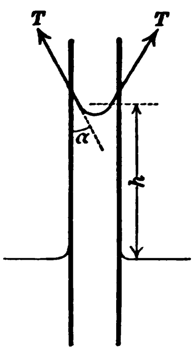


FIG. 53.

The width of the film around the tube is $2\pi r$ centimeters, and the total surface tension in the direction indicated is $2\pi r T$; the vertical component is $2\pi r T \cos\alpha$.

The weight of the liquid supported is $\pi r^2 h d g$, hence for equilibrium we have

$$\pi r^2 h d g = 2\pi r T \cos\alpha \tag{169}$$

whence

$$h = \frac{2T}{r d g}\cos\alpha \tag{170}$$

For clean water on clean glass, the angle of contact is approximately zero, and

$$h = \frac{2T}{r d g} \tag{171}$$

For mercury, α is about 132°; h is negative, and the surface is depressed.

When we consider that the surface tension T, *decreases* with *increase* of temperature, it is seen that the above formula accounts for the inverse relation between capillary action and temperature.

For the elevation between two parallel plates distant u from

each other the computation is similar to that above. The elevation or depression is

$$h = \frac{2\,T\cos a}{u\,d\,g} \tag{172}$$

or one half as great as for a tube of diameter u.

If two plates be joined at one edge and inserted in the water, the liquid rises high along the line of contact and falls off as the plates separate. The upper line of the fluid takes the form of an hyperbola, as shown in Fig. 54.

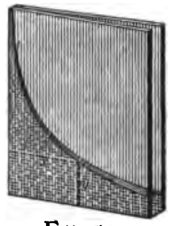


FIG. 54.

88. Solution. Closely allied to the phenomena of capillarity and surface tension are the phenomena of solution. Ostwald defines solutions as "homogeneous mixtures which cannot be separated into their constituent parts by mechanical means." Gases have unlimited power of solution. One gas dissolves in another in all proportions so long as they do not unite chemically, and the homogeneous mixture manifests the sum of the properties of the two constituents.

Liquids dissolve gases without exception, although the readiness with which such solution occurs, varies greatly with the nature and condition of the substances. In a true solution of a gas in a liquid, the gas may be entirely removed from the liquid by diminishing the pressure or by raising the temperature, and in such solutions the quantity of gas dissolved by a given mass of liquid is proportional to the pressure to which the gas is subjected. Examples of this sort of solution are found in solutions of carbon dioxide, of air, or of ammonia gas in water. In other cases as in the solution of hydrochloric acid gas in water, the dissolved gas is not entirely removed from the liquid, and it is assumed that a chemical change has resulted.

The solution of one liquid in another occurs in many cases, but is dependent upon the nature of the substances. Here also there are two distinct classes of solution. Thus some liquids, as alcohol and water dissolve in all proportions forming a homogeneous mixture. Ether and water, on the other hand, dissolve in each other, *but in limited proportions.* Thus water will

dissolve about ten per cent of ether, but if more ether be added the excess remains undissolved. Ether will dissolve about three per cent of water, but beyond this the water remains separate from the ether.

A third division contains those liquids that do not dissolve in each other at all. This is a relatively small number, since even those liquids that seem insoluble, yet leave traces in the solvent. Thus the fact that water when shaken with the volatile oils, retains the characteristic odor of those oils, seems to show that even here solution has occurred in slight degree.

Again mixtures of insoluble liquids, as water and the fixed oils, may be made by the addition of some such substance as gum acacia or gum tragacanth, in which the oils are broken up into exceedingly small globules that float in the water. Such mixtures are called *emulsions.* Milk is a natural emulsion. Some emulsions separate on standing or when subjected to mechanical action, as seen in the separation of cream from milk.

89. Solution of Solids. Many solids when immersed in a liquid gradually disappear and form a new homogeneous liquid. The solid is said to dissolve in the liquid, and the new liquid is called the *solution.* A liquid that dissolves a solid is called the *solvent.* Many salts are soluble in water. The quantity of a substance in solution may vary from zero up to a certain limit, beyond which the solution has no further action upon the substance in question. Such a solution is said to be *saturated.* The amount of a solid that may be dissolved in any solvent varies with the temperature. If the temperature of a saturated solution be changed, either some of the dissolved solid separates out or more of the undissolved solid goes into solution. Generally a solvent will dissolve more of a solid when hot than when cold although there are exceptions to the rule. Thus the solubility of sodium sulphate *increases* with the temperature up to 33°C., but beyond that temperature its solubility *decreases.*

If a solution, either by evaporation or reduction of temperature, be made to contain more than its normal quantity of a solid, it is said to be *supersaturated.* If a particle of the undissolved solid be dropped into the supersaturated solution, the excess of solid in the solution crystallizes out at once.

90. Free Diffusion of Gases. Dalton's Law. Let a tall glass jar be inverted and filled by upward displacement with illuminating gas, and placed upon a similar jar filled with air, with the mouths of the jars together. We shall thus have the two jars filled with separate gases, and since the lighter gas is on top, no mingling of the gases can be produced by the action of gravity. If now the jars be left in position for a quarter of an hour we shall find, on testing the contents with a lighted splinter that there is an explosive mixture of illuminating gas and air in each jar. This is called diffusion of gases, and this result can be explained only on the hypothesis that the molecules of the gases are in motion and that they have therefore wandered through the entire space, the heavier gas rising into the upper jar and the lighter gas descending into the lower jar.

After the gases are uniformly diffused it will be found that the pressure exerted by the mixture is the sum of the pressures exerted by its constituent parts. Thus if we allow 10 volumes of illuminating gas and 15 volumes of air each at atmospheric pressure p, to diffuse uniformly, without change of temperature, through a space of 25 volumes, then, according to Boyle's law, the air would exert a pressure of $15p/25$ and the gas $10p/25$, and their combined pressures would equal that of the atmosphere outside. This important relation was first established by Dalton and is known as *Dalton's law*. It may be stated as follows: *A mixture of two or more gases having no chemical action upon each other, exerts a pressure equal to the sum of the pressures which would be exerted by each of the constituent gases separately if allowed to fill the containing vessel alone at the given temperature.*

It thus appears that each gas behaves as if no other gas were present, the only effect being a slightly diminished rate of diffusion, owing to the mutual molecular collisions. In general the properties of such a mixture are found to be the sum of the properties of the various gases composing the mixture.

91. Diffusion of Gases through Porous Partitions. Atmolysis. A tube, Fig. 55, is partly filled with water, and the right arm closed with a porous cup. Over the porous cup is lowered an inverted beaker filled with hydrogen or illuminating

gas. The water sinks in the right arm and rises in the left, showing an increase of pressure in the cup. Instead of the beaker let a large glass cylinder open at both ends be placed upon a cork on the right arm and the cylinder filled with carbon dioxide which will displace the lighter air, and surround the clay cell. In this case the water rises in the right arm and is depressed in the left, indicating a decrease of pressure in the porous cup. The explanation of these two experiments is very simple. In the first, the lighter illuminating gas diffuses *inward* more rapidly than the air diffuses *outward* and an increase of pressure in the cup results. In the second case the air inside the cup, being lighter than the carbon dioxide, diffuses *outward* more rapidly than the carbon dioxide diffuses *inward* causing a reduction of pressure in the cup.

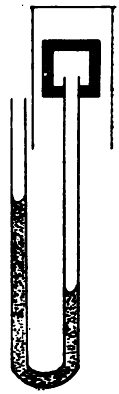


FIG. 55.

The differences in rates of diffusion for different gases have been utilized for separating a gaseous mixture into its constituent parts. Thus if we pass a mixture of hydrogen, nitrogen and oxygen through a porous tube made from the stems of clay tobacco pipes, and maintain a vacuum about the outside of the tube, we shall find that the hydrogen, being the lightest, will diffuse most rapidly, through the walls of the tube, leaving the nitrogen and oxygen behind. Of course some nitrogen and some oxygen escape also, but the mixture transmitted by the tube is relatively richer in the heavier constituents as indicated by equation (162). Rayleigh and Ramsey were able by this means to separate argon from atmospheric nitrogen. This process of separation of gases was first used by Graham, and was called by him *atmolysis*.

92. Diffusion of Gases through India-rubber, and through Red-hot Metals. In 1831 Mitchell observed that toy balloons made of india-rubber collapsed much sooner when inflated with carbonic acid than when filled with air or even with hydrogen. Graham, who studied the phenomenon found that while one volume of nitrogen would pass through a sheet of india-rubber in a given time, 2.56 volumes of oxygen, 5.5 volumes of hydro-

gen and 13.58 volumes of carbonic acid would pass through in the same time. Also the speed with which gases pass through rubber increases very rapidly with increase of temperature.

These remarkable facts do not seem to be connected in any intimate way with the transmission of gases through porous partitions. No simple relation connects the densities of the gases with their speeds of diffusion as in the case of porous septa, and the process of transmission appears to be an entirely different one. The most probable explanation seems to be that the rubber is capable of absorbing and retaining certain amounts of the various gases, with which it comes in contact, the amounts increasing rapidly as the pressure increases. After the surface layer of the rubber has become saturated as it were, with the gas in question, this condition is then passed on from layer to layer of the rubber, until the outside layer is reached. Here, since the pressure is less, the rubber is unable to retain all the absorbed gas and some of it escapes into the adjacent space.

Red-hot metals transmit gases with great facility. The poisonous carbon monoxide passes freely through cast iron at red heat, and from the red-hot cast iron coal stove this deadly gas leaks into the room, almost as water from a sieve. Hydrogen diffuses through a red-hot platinum tube, whose walls are 1.1 mm. thick, at the rate of *490 cubic centimeters per minute, for every square centimeter of surface.* Silver at high temperatures transmits oxygen readily. A glowing tube of palladium, through which is carried a mixture of CO and H separates these gases completely, the hydrogen being transmitted and the carbon monoxide being retained. It thus appears that glowing platinum and palladium act as "semi-permeable membranes" for certain gases, just as certain solid substances do for liquids, in that they allow some substances to pass freely and refuse transmission to others. This peculiarity is of great importance in osmotic phenomena as we shall see later. In all cases of such transmission through rubber or glowing metals, we seem to have to do with a species of solution of the gas on the one side of the partition and of evaporation of the gas from the other side. The same process seems to account for similar behavior in the case of gases and liquid films.

93. Free Diffusion of Liquids. If two liquids that do not react chemically upon each other, be left in contact with each other they will of themselves begin to mingle at once and continue until they form one homogeneous liquid throughout. Thus if a solution of copper sulphate be placed in the bottom of a small jar, and carefully covered with distilled water so the line of separation is well defined and the jar be left undisturbed for a few days, we shall see that the blue color of the copper sulphate has risen into the clear water above and that the line of demarcation is no longer sharp between the liquids. The color of the copper salt at the bottom has also become slightly less dense than at first, and the two liquids seem tending toward a uniform color.

This process is called *diffusion* and while resembling the related phenomenon in gases, its progress in liquids is exceedingly slow. For example, if the jar containing the copper sulphate in the above example be made a meter high, the lower half filled with the solution and upper half containing pure water, it would take more than ten years for the solution to assume a uniform color throughout. If the jar were one centimeter high, it would require about ten hours, the time for uniform diffusion varying as the square of the length of the liquid column.

The speed at which a given solution will diffuse through the pure solvent depends upon the nature of the salt and of the solvent, upon the temperature, and to a slight degree upon the strength of the solution. From extended experiments it has been found that those salts having the highest electrical conductivity, have also the highest velocity of diffusion.

94. Diffusion through Membranes. Osmosis. Crystalloids and Colloids. As we have seen, if two solutions of different strength be brought into contact, a condition of equilibrium can not, in general, be maintained. A movement of the dissolved substance sets in, *from the concentrated to the dilute* solution, and continues until it is uniformly distributed throughout the liquid. If however, we enclose the solution in a vessel fitted with a manometer tube, and provided with a bottom of some porous substance as parchment or animal membrane, and immerse the whole in pure water we shall find that equilibrium is possible.

The porous membrane refuses transmission to the molecules

of the dissolved substance, while allowing the solvent to enter freely. As a result the solvent crowds in through the membrane and creates an internal pressure, as shown by the rise of liquid in the manometer. This crowding in of the solvent continues until the pressure reaches a definite value depending upon the strength of the solution. After this the tendency of the molecules of the solvent to enter the cell seems to be balanced by the internal pressure and equilibrium ensues.

This unequal diffusion through porous septa is called *osmosis,* and the membrane is termed a *semi-permeable membrane.* The limiting pressure beyond which no more of the solvent enters the cell, is called the *osmotic pressure* for the substance, *at that temperature,* and *for that concentration.*

The phenomenon may be illustrated by the following experiment. A conical vessel attached to a long tube is closed at its larger end by a piece of bladder or parchment, firmly tied on. When the vessel is filled with alcohol to the lower end of the tube, and immersed in a vessel containing water as shown in Fig. 56, the liquid in the tube rises to a considerable hight above the level of the water in the outer vessel.

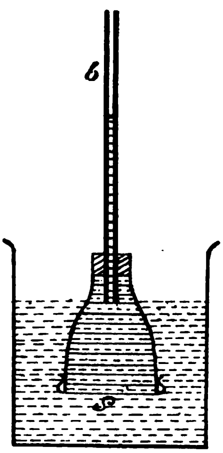


FIG. 56.

Through the experiments of Pfeffer it was discovered that the best results are to be obtained by attaching the tube to a closed clay cell, the pores of which are filled with a precipitate of copper ferrocyanide. The precipitate is pervious to water but impervious to the dissolved substance. With such a cell, filled with a 3.3 per cent solution of potassium nitrate, Pfeffer obtained an osmotic pressure of 436.8 centimeters of mercury, *or more than 5.7 atmospheres.*

Those substances which pass through animal membranes most readily, such as mineral acids, and neutral salts are generally known in the crystalline form and hence have been called *crystalloids.* Substances like gums, tannin, albumen, starch, etc., which are amphorous in character and do not pass through so readily are called *colloids.* The effects of crystalloids when dis-

solved in water, are very marked in the changes produced in the properties of the solvent. Thus any salt dissolved in water ***diminishes its vapor pressure, lowers its freezing point and raises its boiling point.*** Colloids on the other hand, when dissolved in water produce scarcely any such effects. Colloidal solutions in general, represent loose mechanical mixtures from which in many cases the substance held in solution may be precipitated by a slight trace of acid or alkali. When mixed with small quantities of water the colloids form jellies, in some of which the structure is so coarse as to be visible under the microscope. This is notably so of the colloidal solutions of the salts of gold, in which the suspended particles of gold form the objects for ultramicroscopic vision by means of transverse illumination. While many of these colloidal jellies transmit crystalloids almost as readily as pure water, they offer great resistance to the diffusion of other colloids.

95. Osmotic Pressure. We have seen that the entrance of water through the semi-permeable membrane into the osmotic cell may be prevented by subjecting the enclosed solution to sufficient pressure. This pressure is called the *osmotic pressure,* for the substance in question, and is of great importance since the properties of the solution, such as its vapor pressure, its boiling and freezing points are immediately calculable as soon as this pressure has been determined. Osmotic pressure is also intimately related to the transmission of fluids in the living cell of plant or animal tissues. Thus it has been shown that such a cell when placed in concentrated salt solutions, has its liquid contents diminished by the removal of water; if placed in a solution whose osmotic pressure is less than that of the cell, the cell and its contents are distended by the addition of water.

Oil globules of extreme smallness, floating in water tend more readily to pass bodily through the pores of animal membranes, if some alkali be mixed with the water, since they have a soapy covering and are thus passed through the pores like so much water. (Daniell.)

The following conclusions concerning osmotic pressure are stated by Ostwald as reasonably well determined for dilute solutions:

(*a*) The osmotic pressure depends upon the nature of the substance.

(*b*) The osmotic pressure is proportional to the concentration of the solution, or inversely proportional to the volume in which a definite mass of the dissolved substance is contained.

(*c*) The pressure for a given concentration is proportional to the absolute temperature.

(*d*) Quantities of dissolved substances which are in the ratio of the molecular weights of these substances exert equal pressures at equal temperatures.

(*e*) The pressure is independent of the nature of the membrane provided the membrane is impervious to the dissolved substance.

96. Dialysis. The division of substances into *crystalloids* and *colloids* has already been mentioned. A characteristic property of colloids is, that while they pass through porous septa with difficulty *themselves,* they offer no marked resistance to the diffusion of *crystalloids,* but are more or less impervious to other colloids. A result of this property is, that if a mixture of crystalloid and colloid substances be separated from pure water, by a colloidal membrane of a different kind, the crystalloids will soon diffuse into the water while the colloids will remain behind.

This principle has been applied to the separation of crystalloidal poisons from a heterogeneous mass of organic matter in which their presence is suspected. The apparatus as employed by Graham, consists of a hoop, over one side of which has been stretched a piece of bladder or parchment paper, put on wet and held in place by a string. This is slipped inside a second hoop thus forming a shallow dish with a colloidal bottom. This is now floated in a vessel containing distilled water and the substances to be examined are placed in a thin layer on the membranous bottom. If crystalloids be present they will diffuse through into the distilled water, while the colloids are left behind. The water in the lower vessel may then be examined for the suspected poisons by ordinary analysis. The floating dish is called a *dialyzer,* and the process is termed *dialysis.*

PROBLEMS

1. How high will water stand above hydrostatic level in a tube 0.057 cm. in diameter, assuming the angle of contact to be very small?

2. How much will the level of mercury be depressed in a glass tube 0.067 cm. in diameter, the surface tension being taken as 540 dynes per cm. and the angle of contact as 135°?

3. What is the pressure within a soap bubble 12.5 cm. in diameter if the tension of the liquid be taken as 80 dynes per cm., and what is the total perssure exerted by the film on the gas within.

ACOUSTICS

CHAPTER IX

NATURE OF SOUND

97. Definitions. Acoustics is that branch of physics that is devoted to the study of sound and its properties. It differs from other divisions of physics in this respect, that the cause of the phenomena is definitely known, while in Light, Heat and Electricity, the cause is assumed.

In common language the word sound is used in two distinct senses. It may mean the sensations reported to the brain by the auditory nerves, or it may refer to the external cause of those sensations. The psychologist and the physiologist are concerned with the phenomena of sense perception. The physicist is interested in the external disturbance *producing the sense perception*; the conditions of its origin, its mode of propagation, and the variations in the nature of the disturbance corresponding to certain differences in the sensation produced. Accordingly sound is defined by the physicist as *that form of vibratory motion which may be perceived by the ear.* According to this definition sound may exist entirely independent of an ear to perceive or a brain to comprehend.

98. Origin of Sound. Sound originates in a vibrating body. "Sound and movement," says Blaserna, "are so correlated that one is strong when the other is strong, one diminishes as the other diminishes, and the one stops when the other stops." A guitar string plucked aside and released gives a musical note, and at the same time seems to spread out into a broad band with a hazy outline, which diminishes to the original size of the string as the sound dies away. The tremulous motion of a bell, may be perceived by placing the hand upon it while it is sounding. A long glass tube if grasped by the middle and rubbed

with a moistened cloth, gives forth a soft musical note, while the vibratory motion may be plainly felt by the hand The air column in an ordinary tin whistle is thrown into vibratory motion when the whistle is blown. The whistle may even be blown by water and the instrument will give forth a soft clear note, produced by the vibration of the stream of water.

99. Wave Motion. In Article 29, we have seen that a simple harmonic motion compounded with a uniform motion in a straight line produces a "sine curve." Such a curve may be regarded as arising from a series of equidistant particles arranged along the axis $M_0 M_2$, each executing simple harmonic motion at right angles to this axis, but having a common difference of phase; that is, each particle to the right of M_0, crosses the axis going in the positive direction, 1/16 *of a period later than its left hand neighbor.* The result of such a system of moving particles is the sinuous form, known as a *sine curve* or *simple wave.* During the time T, required for any particle to describe a complete vibration, the disturbance will have run to the right, the distance from M_0 to M_2; so that when M_0 begins its *second vibration,* M_2 begins its *first.* Since M_2 marks the extreme advance of the wave disturbance at this instant, it may be said to lie on the *wave front.* A wave front is the continuous locus of all points in the same phase of vibration. The distance $M_0 M_2$ is called a *wave length,* λ, and is related to the velocity of propagation V, and the time of vibration of the particle T, by the equation

$$\lambda = V T = \frac{V}{n} \tag{173}$$

where n denotes the *frequency* or the *number of vibrations,* made by a particle *in one second.*

In the case considered the particles are conceived to vibrate in paths at right angles to the line of propagation of the wave; such a wave is termed a *transverse wave.* Examples are seen in the waves produced in a soft rope when one end is vibrated quickly and regularly in a vertical line.

If the particles vibrate back and forth *in* the line of propagation of the wave, the wave is called a *longitudinal wave.* Such a wave is produced by suspending a long spiral coil of brass

wire between two fixed supports and producing a compression in it by forcing the spirals at one point closer together. The compression runs the entire length of coil.

100. Characteristics of Wave Motion. The fundamental characteristic in wave motion is the continuous handing on from point to point in a medium of a periodic disturbance maintained at the source. Such a disturbance produces a series of waves which follow each other at definite intervals and which constitute what is known as a wave train. It is to be observed that wave motion *transfers energy* from one point to another, by means of the *motion of the particles of the medium.*

The individual particles oscillate about their positons of rest, while the *wave form* runs forward. The distinction between the motion of the particle and the motion of the wave is of fundamental importance. It may perhaps be best illustrated by the motion of a wave passing over a field of grain. The wave form runs forward while the individual head of grain simply swings back and forth in a plane parallel with the line of propagation.

Other characteristics of wave motion may be noted as follows:

The disturbance requires time to travel from one point to another.

A medium is required for transmission of the disturbance.

Waves are reflected on meeting an obstacle, the angles of incidence and reflection being equal.

The direction of the wave is changed, that is, the wave is refracted, on entering a medium in which the speed of propagation is different.

Two systems of waves may be added together so as to reenforce each other, if crest meet crest and trough meet trough; or they may be added so as to annul one another and produce rest, when the crests of one system meet the troughs of the other. This is known as *the interference* of wave systems.

Waves, on meeting small obstacles, bend round corners and, uniting with the waves starting from the obstacles, form a series of points of maximum and minimum disturbance. This phenomenon is known as *diffraction.*

Two waves may differ from each other in *amplitude,* in *wave length,* or in *wave form.*

101. Characteristics of Sound. A sounding body transmits its vibrations to the surrounding medium and a system of sound waves is the result. The waves of sound are longitudinal in character, the vibrations of the particle taking place in the line of propagation. A comparison of the characteristics of wave motion with those of sound reveals the following points of similarity.

Sound requires time for its transmission from point to point, in other words sound travels with a definite velocity, which depends upon the nature and condition of the medium. Examples of this finite velocity of sound are seen in the interval elapsing between the flash of a distant gun and the report of the discharge; or between the appearance of the puff of steam and the sound of the whistle of a distant locomotive, or in the case of the distant woodman who has his axe raised for a second blow before the sound of the former one reaches us.

Sound requires a continuous, elastic and ponderable medium. Sound is not transmitted in a vacuum. An electric bell suspended by rubber cords from the inside of a bell jar is silenced on the exhaustion of the air from the jar, although the clapper may still be seen to be moving. On readmitting the air the sound is heard again. Unless the medium be elastic the energy originally imparted to the medium by the vibrating body is largely lost in heat, as in the case of impact between inelastic solids. The medium must also be ponderable, *i. e.*, it must have weight. Sound is a material phenomenon. It is produced by the vibrations of material particles, and requires a material medium for its transmission. Liquids transmit sound better than gases, and solids, better than either.

Sound is reflected. Whispering galleries are simply large rooms in which the curved surfaces of the walls or roof serve as mirrors for the concentration of the sound waves at particular points. Speaking tubes are devices for preventing the dissipation of the sonorous energy, by confining it to a certain narrow space, through which it passes by repeated reflections from the sides of the tube. Echoes are examples of the reflection of sound by the walls of buildings, by hills or forests.

Sound may be refracted. A large lens-shaped balloon of collodion film, when filled with carbon dioxide acts as a con-

densing lens for sound waves, and brings them to a focus by making the convex wave fronts concave. Beyond the focus the fronts again become convex outward, and the intensity of the sound diminishes.

Sound waves may be made to interfere. An illustration of this fact is the peculiar throbbing or beating effect produced when two tuning forks of slightly different frequency, are sounded together.

102. Sound Waves Longitudinal. If a bell be struck or a tuning fork be put in motion, the layers of air surrounding the vibratory system are thrown into vibrations of the same period, amplitude and form as the vibrations of the sounding body. This means that when the prongs of the tuning fork or the sides of the bell swing *outward* the layers of air, are crowded together, producing a *condensation.* Since in a gas, all the particles are free to move, this condensation is promptly handed on to the next layer of air particles, and by them to the next, and so on. Each layer swinging *outward* in the same manner as the sounding body, only *a little later in time.* On the *inward* swing of the sounding body the layers of air immediately surrounding the body *are rarefied* by the receding body, and the pressure of the air from without promptly fills the rarefied space, only to produce a *rarefaction* where the *adjacent layers had been.* In this way the rarefaction is handed on from point to point, as the counterpart of the *condensation* that preceded it. On the next *outward* swing of the body a second condensation is produced followed by a second rarefaction, and the phenomenon continues while the motion lasts. Now since the condensation and rarefacation each occupy one half a vibration in their production, it follows that *they together constitute the two parts of the sound wave,* and that the distance from one condensation or rarefaction to the *next* condensation or rarefaction is a *wave length,* λ, of the sound in question. Also in the condensation the motion of the particles is *forward,* and in the rarefaction it is *backward,* the wave running its own length during the time of a single vibration of the particles. Since these vibrations are *in* the line of propagation of the disturbance it follows that *sound waves are longitudinal.*

103. Fundamental Differences. Loudness of Sound. Two sounds as perceived by the ear may differ in three respects. They may differ in *intensity*, in *pitch* or in *quality*. These three fundamental differences in sense perception owe their existence to fundamental differences in the vibratory movements producing the sensations. It has already been noted that two waves may differ from each other in *amplitude*, in *wave length*, or in *wave form*. It will be shown later that these three fundamental differences in wave motion determine the three fundamental characteristics of sounds as perceived by the ear.

Loudness or *intensity* of sound denotes in rough measure the kinetic energy of the vibrating medium as reported by the ear. Obviously in so complex a phenomenon as sense perception no rigid measure of the absolute intensity of a sound is possible. Not only does the same sound appear of different intensity to two different observers, but to the same observer on different days and in different states of health and nerve tension. Comparisons of the relative intensity of two or more sounds of the same general character are possible, however, and show that the intensity of sound depends essentially upon three things.

(*a*) *Upon the amplitude of vibration of the sounding body.* We have seen that two waves of the same frequency may differ in the relative excursions executed by the vibrating particles. From elementary considerations it may be seen that if the excursions of the particle be increased, the velocity of the particle must be proportionally increased, since the vibrations in each case are executed *in the same time*. The velocity of the vibrating particle is therefore proportional to the amplitude. But the kinetic energy of the particle is proportional to the *square of the velocity*, and consequently to the *square of the amplitude*.

(*b*) *Upon the area of the sounding body.* The effect of area in modifying the loudness of a sound produced by a given body, is essentially a question of transference of energy. A tuning fork when once set in motion, possesses a definite amount of kinetic energy. If the fork be held in the hand, the sound produced is faint, but continues for several seconds. If the stem of the fork be placed firmly upon the top of a table the note becomes loud and strong and soon dies away. In the latter case the top of the table is thrown into co-vibration with the fork, and

the volume of air that is energized by this means is greatly increased. The store of energy is spent in the production of sound in a much shorter time. In hydrogen the sound of the fork is very weak, but the motion continues much longer.

(*c*) *Upon the distance from the sounding body.* The intensity of sound varies inversely as the square of the distance from the sounding body. This is known as "the law of inverse squares," and may be demonstrated as follows: Suppose a single sharp sound has been produced at a point O, as a center. This represents a definite amount of energy. Now at any instant this energy is resident on a spherical shell of radius r, and of area $4\pi r^2$ cms. If we call the intensity of sound I, the kinetic energy possessed by unit area of this shell, then the total energy is $4\pi r^2 I$. An instant later this energy has been transferred to a shell of radius r_1, on which the intensity is I_1, hence the total energy on the shell is $4\pi r_1^2 I_1$.

Therefore

$$4\pi r^2 I = 4\pi r_1^2 I_1 \tag{174}$$

whence

$$\frac{I}{I_1} = \frac{r_1^2}{r^2} \tag{175}$$

or the intensity of sound varies inversely as the square of the distance from the sounding body.

From a physical standpoint the intensity of a sound is best defined as the *quantity of energy passing in unit time through unit area of a surface placed at right angles to the direction of the sound.* From this definition accurate measurements of the intensity are possible, but their realization is difficult and involves considerations beyond the limits of an elementary text.

CHAPTER X

VELOCITY OF SOUND

104. Experimental Determinations. The problem of determining experimentally the velocity of sound in air presents peculiar difficulties. The nature of the problem demands that the measurement be made in free, open air, where the disturbing effects of winds, which vary both in magnitude and direction, and of local differences in temperature, are unavoidable and beyond the control of the observer. To these are added the errors of observation attendant upon measurements depending upon sense perception.

If the measurements are made by one observer who notes and records the time of seeing a flash, and of hearing the report of a distant gun, then each observation is affected by the errors due to the reaction periods of both sight and hearing. By this is meant the time required for any observer to see and record the flash, and to hear and record the sound. The reaction period for sight is different from that for hearing. The error for each operator is termed his "personal equation." Not only do these errors vary with different operators, but they vary with the varying conditions of health of the same operator on different days, and with the intensity of the sound. No allowance is made for the time occupied in transmitting the flash, since for small distances this time is practically zero.

A study of the various errors just mentioned has influenced the selection of methods of observation. If cannon be used as the source of sound, the method of "reciprocal firing," whereby the sound is transmitted first in one direction and then in the other between two stations, has been used to eliminate in large measure the errors due to the wind. This, however, requires two observers, and introduces errors arising from differences in "personal equation" which are difficult to determine and cannot in general be wholly eliminated by exchange of observers. In the

experiments of Stone in 1871, two operators about three miles apart, recorded the time of hearing the report of a cannon placed some 640 feet distant from the first operator. On exchanging operators the error from "personal equation" was reduced to 0.02 sec.

The mean of Stone's results, reduced to 0°C., was 1090.6 feet per second, or 332.4 meters per second.

More accurate determinations of this important constant, made by Violle and Vautier give for the velocity of sound in air at 0°C., the value

$$V = 331.36 \frac{m.}{sec.}$$

and this value is now generally accepted as the best result yet obtained.

105. Experiments of Regnault. Between the years 1862 and 1866 Regnault carried on an exhaustive series of experiments for the determination of the velocity of sound both in the open air and in the water and gas pipes of Paris. In his researches Regnault made use of an automatic recording apparatus, by means of which an electric current was broken at the instant of firing the gun, and the interruption of the current was recorded upon a smoked paper carried upon the drum of a chronograph. At the receiving station the sound wave entered a wide cone, at the smaller end of which it impinged upon a thin rubber membrane, and setting it in motion broke a second electric current, and so completed the record upon the cylinder of the chronograph. By this means it would seem that the difficulties of personal equation were entirely obviated, but it was found that the membrane itself required time to receive and record the sound wave. The motion of the air particles can not be imparted to the membrane instantly, and so a delay is caused in making the record, which is not constant but increases as the sound grows more faint. Regnault made experiments to determine the amount of this error, and allowed for it in his computations.

In his experiments upon the velocity of sound in tubes, Regnault arrived at the following conclusions:

(*a*) In cylindrical pipes the intensity of the sound wave decreases with the distance, and more rapidly in small tubes than in large ones.

(*b*) The velocity of sound diminishes with the intensity. Loud sounds travel faster in tubes than faint ones.

(*c*) The velocity of sound in pipes increases as the diameter of the pipe increases, tending toward a limit in very wide tubes.

(*d*) The velocity is independent of the pressure, and of the mode of production of the sounds.

Regnault gave as the result of his investigations, after all corrections had been applied, the value for a faint sound in a very wide tube, at 0°C.

$$V_0 = 330.6 \frac{m.}{sec.}$$

106. Theoretical Velocity of Sound. From purely theoretical considerations concerning the condition of the medium through which the sound wave passes, Newton, in 1686, developed a formula for the velocity of sound,

$$V = \sqrt{\frac{e}{d}} \tag{176}$$

where d is the density of the medium and e its coefficient of volume elasticity. This formula applies directly in the case of liquids and solids, while for gases, it leads to erroneous results unless there be applied certain corrections, the nature of which will appear later.

In the application of this formula to gases, it is to be shown that the coefficient of volume elasticity e, is, for ordinary sounds, equal to the pressure to which the gas is subjected. Thus let P, V, and d represent the pressure, volume and density, respectively of a given mass of air. Let the pressure be increased by a small increment dp, and as a result of this increase, let the volume be diminished by a small amount dv, *the temperature remaining constant*. Then by Boyle's law we have

$$\frac{P}{P+dp} = \frac{V-dv}{V} \tag{177}$$

or

$$\frac{dp}{P+dp} = \frac{dv}{V} \tag{178}$$

whence

$$\frac{dp}{\frac{dv}{V}} = P + dp \tag{179}$$

but the left hand member of this equation is by definition the coefficient of volume elasticity, e, (Article 51), hence $P + dp = e$.

For ordinary sounds the change in pressure dp, produced in the condensation or rarefaction, is negligible as compared with the atmospheric pressure, and hence may be disregarded. Under this assumption our formula becomes,

$$V = \sqrt{\frac{P}{d}} \qquad (180)$$

where P is the reigning barometric pressure, and d the corresponding density of the air.

This would indicate that the velocity of sound is independent of the intensity, and this assumption seems to be sustained by observation in the case of all sounds where the change of pressure dp, produced by the condensation is negligible in comparison with the barometric pressure, P. In disturbances where dp is not negligible, the velocity is represented by the formula

$$V = \sqrt{\frac{P + dp}{d}} \qquad (181)$$

and increases as the intensity increases. In loud sounds, as those produced by explosions or the discharge of cannon, the change in pressure dp, may even exceed P. Captain Parry, who made a series of measurements upon the velocity of sound in the arctic regions, relates that the report of a cannon was frequently heard by a distant observer, before the command to fire. Other observers have repeatedly found that very loud sounds travel more rapidly than those of ordinary intensity.

107. Application of Newton's Formula. It has been said that Newton's formula $V = \sqrt{\frac{P}{d}}$ does not apply directly to the case of gases. If the values of P and d be substituted in this formula we have for standard conditions, $P = 1{,}012{,}630$ dynes per sq. cm., and $d = 0.001293$ g. per cc. Whence

$$V = \sqrt{\frac{1{,}012{,}630}{0.001293}} = 27{,}927 \frac{cm.}{sec.}$$

This is only 84 per cent of the velocity of sound as determined by experiment. The discrepancy between the theoretical and observed value was recognized by Newton, who sought to account for the difference by means of a number of ingenious suppositions, none of which, however, were justified.

108. Laplace's Correction. It was not until 1816 that the error in Newton's formula was pointed out and corrected by Laplace. In Article 106, it was noted that for small disturbances the coefficient of volume elasticity *e*, in the formula

$$V = \sqrt{\frac{e}{d}}$$

could be replaced by the pressure *P*, to which the gas was subjected.

This deduction was made upon the assumption that Boyle's law would hold for the phenomena. This would mean that the air in which the condensations and rarefactions of the sound wave were produced should remain throughout *at a constant temperature*; or in other words, that the condensations and rarefactions were to occur under *isothermal conditions.*

Laplace pointed out that in the case of a sound wave, where the compressions and rarefactions follow each other at the rate of several hundred per second, there could be no mention of isothermal conditions, since there could be no time for the air to assume a uniform temperature. The coefficient of elasticity of the air, that really does enter the equation for the velocity of sound, is therefore to be determined, under the condition that *no heat is to escape from the gas when it is compressed, and none is to come to it when it is rarefied.* Such conditions for a gas are called *adiabatic conditions,* and the coefficient of elasticity of a gas obtained under such conditions is called *the coefficient of adiabatic elasticity.*

Now it is well known that if a mass of gas be compressed suddenly, its temperature is raised, and the elastic tension of the gas is greater *because of this increase in temperature.* Careful experiment has shown that for air, the coefficient of *adiabatic elasticity* is 1.41 times the coefficient of *isothermal elasticity.*

Hence the pressure P must be multiplied by 1.41 to represent the facts. The corrected formula is therefore

$$V = \sqrt{\frac{1.41\,P}{d}} \tag{182}$$

When the correcting factor is introduced into the computations in the previous article we have

$$V = \sqrt{\frac{1.41 \times 1{,}012{,}630}{0.001293}} = 332.3\ \frac{m.}{sec.}$$

which agrees admirably with the value found by experiment.

109. Correction for Temperature. Since the air is free to expand, an increase in temperature of the air as a whole, will affect the *density* of the air, but will leave the pressure unchanged. If a be the coefficient of expansion for gases, then any bulk of air at 0°C. will have a bulk $(1 + a\,t)$ times as large at t°C. Since the densities are inversely as the volumes, then d_t, the density of the air at t°C., will be given by the equation

$$d_t = \frac{d}{1 + at}$$

Hence the velocity of sound at t°C. will be

$$V_t = \sqrt{\frac{1.41 \times P}{d_t}} \tag{183}$$

or

$$V_t = \sqrt{\frac{1.41 \times P(1 + a\,t)}{d}} \tag{184}$$

If V_0, be the velocity of sound at 0°C., then

$$V_t = V_0 \sqrt{1 + a\,t} \tag{185}$$

If we set $t = 1$°C. and $a = 0.003665$, then the *increase* in velocity for a rise in temperature of 1°C. is

$$V_1 - V_0 = 331.36 \sqrt{1 + 0.003665} - 331.36$$
$$= 0.00183 \times 331.36$$
$$= 0.606\ \frac{m.}{sec.} \text{ or } 23.8\ \frac{in.}{sec.}$$

Hence the velocity of sound increases about 60.6 cm. per second, or 23.8 in. per second for an increase of 1°C.

110. Velocity of Sound in Solids and Liquids. For solids and liquids Newton's formula is applicable at once. For copper the coefficient of elasticity, *e*, is about 12×10^{11} dynes per sq. cm. and $d = 8.8$ g. per cc., therefore

$$V = \sqrt{\frac{12 \times 10^{11}}{8.8}} = 369300 \frac{cm.}{sec.} = 3693 \frac{m.}{sec.}$$

For water the density is 1 gram per cc. and the *compression for an increase in pressure of one atmosphere* is 0.0000499; hence

$$V = \sqrt{\frac{1{,}012{,}630}{1 \times 0.0000499}} = 142500 \frac{cm.}{sec.}$$

$$= 1425 \frac{m.}{sec.}$$

The best experimental determinations of the velocity of sound in water give a mean result of 1435 meters per second.

CHAPTER XI

REFLECTION AND SUPERPOSITION OF SOUND WAVES

111. Huygens's Principle. Suppose a sound wave originate at *a*, as a center of disturbance. The wave will travel outward in space in all directions in the form of a spherical shell. Let Fig. 57 represent the trace on the plane of the paper of a section through this wave shell. At the end of a certain time, the wave will have reached the position *m c n*. This surface, marking the location of all points in the same phase of vibration, is by definition a *wave front*. At an instant later it will have taken the position *m′d n′*, and all points in this new wave front will be vibrating with the same motion possessed *now* by the points on the wave front *m c n*. The disturbance at the center has thus reached any point in *m c n* by the disturbance of all points in the medium through which it has passed. Hence the subsequent disturbance at any point outside the wave *m c n* is to be regarded as the resultant effect produced at that point, of all the wave disturbances originating in the individual points of the wave front *m c n* as centers.

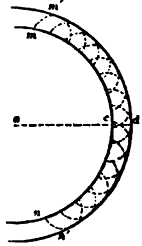


FIG. 57.

According to *Huygens* we are to consider *every point on a wave front as a new center of disturbance, from which waves are propagated outward as from the original center, and the effect at any point external to the wave is to be regarded as the resultant of the combined action of all these elementary waves.*

If we take all the points on the wave front *m c n* as centers and about them describe small circles representing the limits reached by the elementary waves in a very short space of time, then it will be seen that these little waves unite to form a *new wave front,* which is a common tangent or *envelope* to them all. This is to be regarded as the new wave front. Professor Stokes

has shown that the elementary waves destroy each other except at the surface of their common envelope.

112. Reflection of Sound. Sound waves are reflected regularly from any smooth surface of different density from that of the medium in which they are travelling, and the sound seems to come from some point behind the reflecting surface. By a "smooth" surface is to be understood a surface whose inequalities are small in comparison to the wave lengths to be reflected. Thus a brick wall, a hillside or even a row of trees may present a surface that is relatively smooth to the waves of sound, while for the reflection of light waves only the most highly polished surfaces are "smooth."

Tyndall showed that sound waves are reflected not only from solids and liquids but from layers of gas of different density as well. Thus a flame from a fish-tail gas burner and the heated air above it, reflect sound waves to a marked degree. In the same way ascending currents of warm air over parts of the earth unequally heated by the sun, serve to reflect and scatter sound waves, as do also ascending layers of air loaded with water vapor.

In some cases the nature of the sound is modified by the nature of the reflecting surface, as in the case of a sharp clear sound reflected by the separate bars of a picket fence. Here the separate pickets act as independent sources of sound waves and the short sharp sound comes back as a clear musical note of definite pitch.

Echoes are produced when the reflecting surface is at a distance of more than 16.5 meters. The ear can not separate syllables occurring more rapidly than about ten per second. Hence a sound and its echo will be heard separated if the distance to the reflecting surface is greater than half the distance sound would travel in one-tenth of a second.

Applications of the principle of reflection of sound are seen in the speaking trumpet or megaphone, the ear trumpet, the stethoscope, and in the curved surface of the external ear which acts as a reflector to concentrate the sound waves into the auditory canal.

113. Reflection at End of Cylindrical Pipe. A case of special interest is found in the reflection of a sound wave at the end of a tube or pipe in which it is travelling. If the reflections occur in a closed pipe, against the solid end of the pipe, the direction of the wave is reversed, but the nature of the disturbance is not reversed, a condensation is reflected as a condensation and a rarefaction as a rarefaction. This is expressed by saying that at the closed end of a pipe the sound wave is reflected with change of sign in the *velocity,* but without change of sign in the *condensation.*

At the end of an open pipe, however, the result is different. When the disturbance reaches the open end of the pipe the last layer of air having no longer an opposing resistance in front, as it had within the tube, swings outward beyond its position of equilibrium, dragging after it the adjacent layers one after another. A new disturbance thus enters the tube at the open end, in the nature of a rarefaction, and is propagated backward while the original condensation pursues its course in the free air. This is expressed by saying that reflection occurs at the open end of a pipe, *without change of sign in the velocity, but with change of sign in the condensation.* At the open end of a pipe therefore, a condensation is reflected as a rarefaction and *vice versa.* This principle will be found of great importance in the theory of open and closed organ pipes.

114. Superposition of Sound Waves. If two stones be dropped a small distance apart into still water, each is seen to become the center of a system of circular waves, widening in all directions as they run. If the two sets of waves cross each other they are seen to pursue their own way in each case as if there were no other wave on the surface of the water. In general, when two systems of waves traverse a medium simultaneously, while the elevation or depression of either alone relative to the disturbed surface is the same as it would have occasioned if the other were not there, yet the total displacement of the surface relative to the undisturbed level, is at any point and at any instant the algebraic sum of the disturbances which each of the systems would produce separately.

If an elevation of the first system be superposed upon an

equal elevation of the second system, the total hight of the water above the original level will be double that of one of the two waves alone. If an elevation of one system meet an equal depression of the other system the original level is not changed.

In a similar way, sound waves, when coexistent in the same medium, form at every instant composite waves, made up of the separate systems. This superposition of wave systems is termed *interference.*

115. Principle of Interference. If we consider two identical sources of sound, A and B (Fig. 58*a*), such that the vibratory motions in each are at every instant equal and in the same direction, then it is clear that at every point in a plane $P\,P'$, normal to the line $A\,B$ at its middle point, the movements produced by the joint action of the two sets of sound waves will agree at every instant. In other words in this plane the two wave systems meet everywhere *in the same phase.* The *velocity* and the *displacement* will be larger than those from either source separately. The intensity of the sound will therefore be a maximum in this plane.

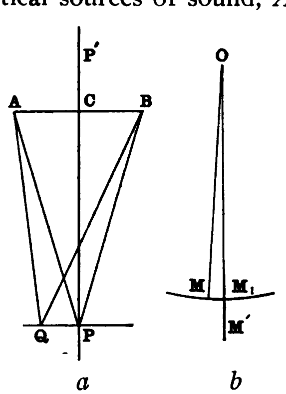


FIG. 58.

In any other plane drawn through AB, the *displacements* and consequently the *intensity* will present a series of fixed maxima and minima. For our purpose it will be sufficient to consider the vibratory motion producing the sound to be divided into half vibrations, equal and opposite; *i. e.,* to consider each wave to be made up of rarefaction and condensation, and the motion of the air particles in these parts to be equal and opposite. If we consider the disturbance from the point O, (Fig. 58*b*), along two lines OM and OM' very near to each other, so that we may assume that the motions have experienced the same conditions in passing from O to M and M', then we shall find the particles in M and M' to be in the *same phase of vibration* if the difference of the distances OM and OM' be an even number of half wave lengths. They will be in *opposite phase* if the difference of the

paths OM and OM' be an *odd* number of half wave lengths. In the first case the velocity and displacement of the two particles at M, and M', will be equal and in the same direction, in the second, they will be equal and in opposite direction.

From the foregoing considerations we see that when sound waves from two identical sources A and B (Fig. 58*a*), meet at a point Q, the velocities will add if

$$AQ - BQ = 2n\,\lambda/2 \tag{186}$$

and will subtract if

$$AQ - BQ = (2n + 1)\,\lambda/2 \tag{187}$$

that is, according as the difference of path $AQ - BQ$, is an *even* or an *odd* multiple of $\lambda/2$.

Hence for maximum intensity the difference in path is an *even* number of half wave lengths, for minimum intensity the difference in path must equal an *odd* number of half wave lengths. In this way two equal sounds may be added together so that at certain points in space they will mutually reënforce each other, and the sound will be very loud; in other points where the vibrations meet in opposite phase, one sound added to another may produce silence.

116. Curves of Maximum and Minimum Intensity. The interference of waves from two identical sources of sound may be shown graphically by the following construction: About the point sources A and B (Fig. 59), describe arcs of circles with equal radii, where the full line arcs denote condensations and the dotted arcs represent rarefactions. The shortest distance between any two concentric arcs of the same kind is therefore some multiple of the distance from one condensation to another condensation, or from one rarefaction to another, *i. e.*, some multiple *of a wave length* λ, of the sound in question. Similarly the distance between a full line arc and the next adjacent dotted arc described about the same center, is *a half wave length.*

From the foregoing considerations it is evident that at all points on the line CP, normal to AB at its middle point, the two sets of waves *meet in the same phase,* condensation from A meeting condensation from B, and rarefaction from A meeting rare-

faction from B as indicated by the intersection of the corresponding arcs. The dotted line CP is therefore *a line of maximum intensity*, and the remaining dotted lines cutting AB, being also loci of particles in the same phase of vibration are *also lines of maximum intensity*. The full lines on either side of CP, drawn

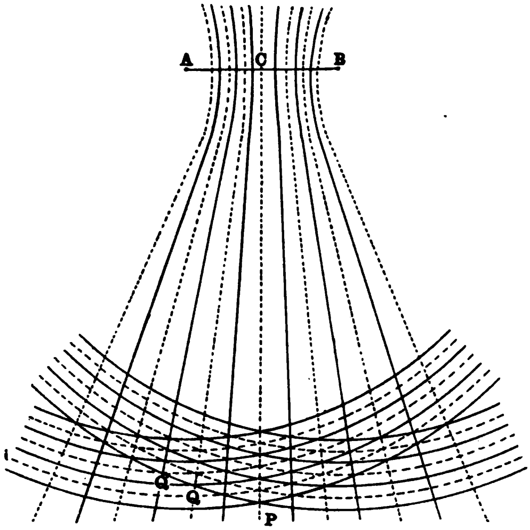


FIG. 59.

through the intersections of rarefactions from A with condensations from B, or *vice versa*, are consequently lines of *minimum intensity of sound*. These curves are all hyperbolae, defined by the equation

$$AQ - BQ = \pm n \lambda/2 \tag{188}$$

where Q is any point whatever, and n is any integer 0, 1, 2, 3, &c. The lines of maximum intensity are given by the even values of n, and those of minimum intensity by the odd values.

We should remember that there is *no loss of energy* due to interference. At the maxima the amplitude is *double*, and the energy *four times* that due to a single source. At the minima, the amplitude and the energy are both zero. Hence the average energy over the surface is *twice* that due to a single source, as it should be.

117. Experiments Illustrating Interference.

(a) Sources Identical.

If a tuning fork be sounded and rotated slowly before the ear, an intermittent or pulsating sound will be heard. At four certain positions, it will be found that the sound of the fork seems to disappear almost entirely, only to reappear again in force on moving the fork. In this case the two identical sources are the two prongs of the fork, which are vibrating in opposite phase as indicated in the arrows—(Fig. 60). As the prongs approach, there is a rarefaction on the outside of each prong at *f* and *g*, and a double condensation on the inside, starting out toward *d* and *e*. These two sets of motions differing in phase by half a period meet along the dotted lines and produce a minimum motion at all points in these lines. Outside these lines the sound may be heard as usual.

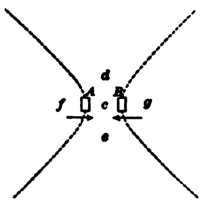


FIG. 60.

To show that this is a true case of interference the fork may be held over a resonator which responds loudly, and rotated till the sound falls to a minimum. On slipping a small cylinder of paper over one prong of the fork, so as to cut off one set of waves, the sound reappears, but disappears again if the paper be removed.

If a large fork or organ pipe be sounded in a large room and an observer walk about the room, certain places will be found where the sound is uncomfortably loud while at others almost no sound is heard. In this case interference occurs between the direct waves from the fork and the systems of waves reflected from the sides of the room.

(b) Sources not Identical.—Beats.

If two tuning forks, tuned to unison and furnished with resonance cases, be sounded together, a full even tone is heard. If now the prongs of one fork be weighted with wax, so as to decrease slightly the frequency of that fork, we shall find that when sounded together, the forks give out a throbbing or beating

tone. If the pellets of wax be made larger the number of beats per second, increases.

In this case the phase difference $\lambda/2$ is not due to the difference in path traversed by the waves from the sources, as hitherto, but has resulted from a difference in the *frequencies* of the two forks. If the forks make respectively 100 and 101 vibrations per second, then the second fork gains one vibration upon the first every second. Under these conditions, if the two forks *started* together in the same phase, they would be in opposite phase at the end of half a second, and in coincidence of phase again in another half second. From this it appears that two forks whose frequencies are m and n vibrations per second will make $m - n$ beats per second.

Two organ pipes of the same pitch, when mounted upon a wind chest, will give loud beats if while they are sounded, a card be slipped slightly over the lip of one of the pipes. A tuning fork mounted upon a resonance case and sounded gives distinct beats if carried rapidly *toward* or *away* from a reflecting wall. The experiment may be rendered more striking by swing-the fork as a pendulum, while sounding, at a short distance from the wall.

CHAPTER XII

MUSICAL RELATIONS

118. Pitch. The pitch of a sound depends upon the vibration frequency of the sounding body. When the number of vibrations per second is great the pitch of the tone is high or *acute,* when small the pitch is low or *grave.* If two sounds are produced by the same number of vibrations per second they are said to have the same pitch, or if sounded together, they are said to be in *unison. Musical sounds* are those which produce a pleasing effect upon the ear and *have a definite pitch.* A noise is a confused mass of sonorous vibration in which the ear is unable to detect any definite pitch. In a musical sense, the pitch of a sound may also refer to the relative position of the sound upon some arbitrary scale of reference adopted by musicians.

The pitch of a sound may be determined by means of an instrument called the siren (Fig. 61). The siren consists of a pasteboard or metal disk, bearing on its circumference a series of concentric circular rows of equidistant holes about 3 mm. in diameter, and mounted on an axis which can be rapidly revolved in front of a nozzle delivering air from a blower. When an opening comes in front of the nozzle the air rushes through, forming a condensation, followed by a rarefaction during the interval in which the air is cut off. In this way is formed a series of regular puffs, which gradually blend into a low musical tone, whose pitch rises as the speed of rotation of the disk increases. The siren tone may be tuned to that of the given note, and its frequency determined from the angular velocity of the disk and the number of holes passing the nozzle in one revolution.

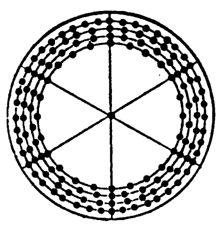


FIG. 61.

119. Musical Intervals. When two tones are sounded together the ear recognizes a certain relationship, or want of relationship between them, dependent upon their relative pitch, but entirely aside from their absolute vibration frequencies. This relationship is termed a *musical interval,* and is expressed as a simple ratio between the vibration frequencies of the two tones in question.

A number of these ratios have specific names in musical nomenclature, arising for the most part from the number of the note in the series. The interval between two notes whose vibration frequencies are in the ratio 1/1 is called *unison*; 2/1, an *octave;* 3/1, a *twelfth;* 4/1, the *double octave;* 3/2, a *fifth,* 4/3, a *fourth;* 5/4, a *major third;* 6/5, a *minor third;* 5/3, a *major sixth;* 8/5, a *minor sixth.*

This completes the list of so-called *consonant intervals,* although the list may be and probably will be extended in the course of time. It is interesting to note that the third, both major and minor, were originally classed among the dissonant intervals, and the minor third was not regularly used until the middle of the eighteenth century.

120. The Diatonic Scale. The rule for consonant intervals extends to combinations of several sounds. In order that three or more tones when sounded together may be concordant it is necessary that their respective intervals not only with the fundamental, but also with each other, should be expressed by simple ratios. Thus when we sound together three notes whose frequencies are as 4 : 5 : 6, there is produced a pleasing effect. This combination of three tones is called a *major triad.* The diatonic scale is built upon three sets of such triads.

The notes of the scale are indicated by the letters *C, D, E, F, G, A, B, c.* These letters may represent the frequencies of the various notes as well.

In the key of *C*, the three major triads are

$$\left.\begin{array}{l} C : E : G \\ G : B : d \\ F : A : c \end{array}\right\} 4 : 5 : 6 \qquad (189)$$

The vibration ratios in terms of C, the fundamental, are obtained as follows:

$$\frac{E}{C} = \frac{5}{4} \text{ or } E = \frac{5}{4}C$$

$$\frac{G}{C} = \frac{6}{4} \text{ or } G = \frac{3}{2}C$$

$$\frac{B}{G} = \frac{5}{4} \text{ or } B = \frac{5}{4}G = \frac{5}{4} \cdot \frac{3}{2}C = \frac{15}{8}C$$

In this way the frequencies of the entire seven notes may be expressed in terms of the fundamental C, the octave c, being set equal to $2C$, and we have the following relations between the various notes of the scale.

Frequency	64	72	80	$85\frac{1}{3}$	96	$106\frac{2}{3}$	120	128
Name	*C*	*D*	*E*	*F*	*G*	*A*	*B*	*c*
Syllable	*Do*	*Re*	*Mi*	*Fa*	*Sol*	*La*	*Si*	*Do*
Ratio	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
Intervals	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{9}{8}$	$\frac{16}{15}$	

The fractions termed intervals are obtained by dividing each ratio by that of the note immediately below it.

From this it appears that in the perfect diatonic scale there are three different intervals, 9/8, 10/9, 16/15. The first two are termed *whole tones,* and the last a *half tone.*

The *minor triad* is composed of three notes whose frequencies are as 10: 12: 15, and the *minor scale* is built upon three of such triads. The vibration ratios for the various notes in terms of the fundamental may be obtained in the same way. It will be found that three addtional notes will be needed, viz., three notes below E, B and A, in order to produce the minor scale.

121. Transposition. In order to accommodate different voices or instruments, it is frequently desirable to change the key-note of the scale from C to some other note in the scale. The vibration ratios would then have to be applied to the *new key note* as a fundamental, and the corresponding frequencies for the several notes computed. If it were desired to begin

the scale with *D,* then on computing the frequencies for the scale it would be found that, besides the key-note *D,* and its octave *d,* the *G* and *B* were right and that the *A* and *E* differed but slightly from the required frequency, but the notes *F* and *c* would be found to be too low in each case. This must be remedied by the introduction of two new notes *F* sharp, and *c* sharp, in order to sing or play in the key of *D.* These two sharps are introduced at the beginning of the staff and form the signature of the key.

122. The Tempered Scale. Since each change of key entails the introduction of new notes, both for major and minor scales, it is apparent that the number of notes demanded for each octave, in order to render a piece of music in any key, would be very greatly increased, so much so indeed, that in the case of an instrument of fixed tone, as the piano or organ, it becomes practically impossible to manipulate so many keys. On this account a compromise system known as the system of *equal temperament,* has been adopted. In this system the whole tones are made all alike, and the half tones are half the whole tones. In other words *there is an equal interval between each pair of consecutive notes.* There are thus added five new tones to the octave, making thirteen tones in all. The common ratio between the frequencies of any tone and the tone next above it is the *twelfth root of 2,* or 1.059. In any instrument tuned to this system the only accurate intervals are the octaves, all the others being slightly false. The fifths are slightly flat and the thirds are too sharp. Music rendered in this system is considered inferior to that played in just intonation. Trained singers and performers upon instruments like the violin or slide trombone are free from the limitations of the system of equal temperament, and in many cases approximate closely the intonation of the diatonic scale.

CHAPTER XIII

VIBRATORY PHENOMENA AND RESONANCE

123. Composition of Vibrations at Right Angles. In the study of vibratory motion, some curious and beautiful results are obtained from combining two simple harmonic motions at right angles to each other. Owing to the rapid motion of sounding bodies the eye is unable to follow them and some special device is necessary. In Fig. 62 are shown two tuning forks *L* and *M*, each furnished with plane mirrors, one set to vibrate vertically, the other horizontally. If now a beam of light from *R* be allowed to fall upon the mirrors successively and be reflected to the screen *S*, when the fork *L* is set vibrating the spot is seen drawn out into a vertical band. Similarly if the fork *M* be set in motion and *L* kept at rest, the spot is drawn out into a horizontal band. If the two forks are vibrated at the same time, the spot is made to follow the motion of the two vibrating systems and traces the characteristic curves shown in Fig. 63.

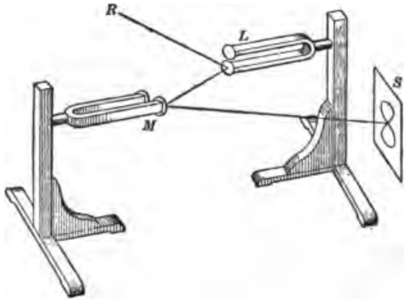


FIG. 62.

If the frequencies of the two forks be in the ratio of 1:1. then the characteristic figure will be an ellipse, having for its special forms the two straight lines. If the tuning of the fork be exact, the figure is motionless upon the screen and gradually decreases in size as the amplitude of the vibration of the forks sinks to zero. In most cases, however, the tuning is only approximate, and the figure takes the successive forms indicated passing from left to right and back again. When the figure has run through

the complete cycle we know that one of the forks has gained or lost one complete vibration as compared with the other. We have thus a method of observing beats optically, and determining the relative frequencies of vibrating bodies with great precision.

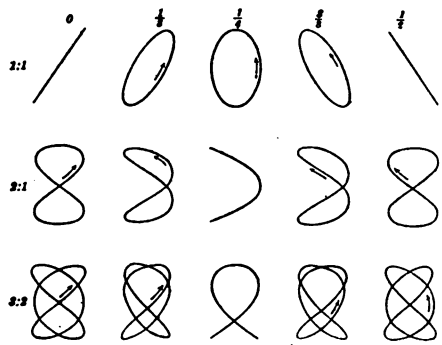


FIG. 63.

In the figure are shown the curves for the intervals 2:1 and 3:2. The characteristic forms for zero difference of phase are given in the first vertical column, those for a phase difference of 1/8 period in the next column and so on. After the figures have reached a phase difference of half a period, the forms recur in the reverse order.

This experiment was first described by the French physicist Lissajous, in 1857, and the curves are known as Lissajous' curves. The method is applicable to the study of any vibrating system upon which a bright point as a minute globule of mercury can be fixed, while the fork with which the system is to be compared, is armed with a lens of low power through which the mercury globule may be viewed by a microscope.

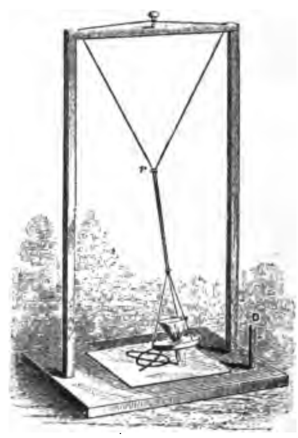


FIG. 64.

The same figures may be obtained by means of the Blackburn's pendulum, shown in Fig. 64. In this apparatus a heavy lead disk carries a funnel filled with sand, ink, or other material for leaving a trace of the motion upon a prepared paper beneath it. The disk is hung from two cords about one meter in length, and over the two cords is slipped a small ring *r*,

by means of which the system is divided into two pendulums of different period, hung from the same support. On setting the disk vibrating it traces the figures characteristic of the ratios represented by the square roots of the lengths of the entire pendulum and the part below the ring.

124. Graphical Method for Lissajous' Figures. Draw two concentric circles (Fig. 65), with radii proportional to the amplitudes, *a* and *b*, of the two harmonic motions, and through their common center *O* draw the rectangular diameters *AB, CD*.

Divide each quadrant of both circles into the same number of equal parts; some multiple of four is usually most convenient. Through the points of division of the circle *AB* draw lines parallel to *CD*, and through the divisions of *CD* draw lines parallel to *AB*. The resulting rectangle of sides $2a$ and $2b$ will contain all the figures arising from any possible combination of two simple harmonic motions of commensurable periods; and the curves will, in general, be tangent to the sides of the rectangle. The center of the circles corresponds to a phase difference of zero between the two components, that is, to $\delta = 0$; and it is taken as the starting point for tracing all curves of phase difference zero or π.

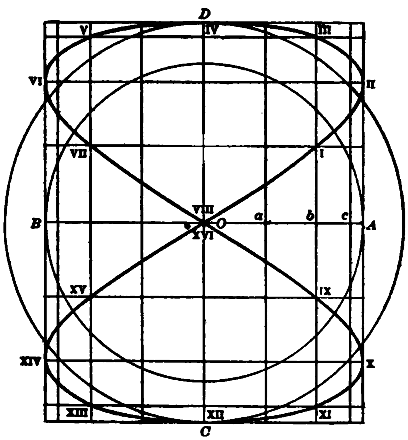


FIG. 65.

If, as in Fig. 65, the circles have been divided into sixteen equal parts, then each point of intersection on the diameter *AB* corresponds to a phase difference of $2\pi/16$ or $\pi/8$, that is, to one-sixteenth of a period. Hence if we start to trace a curve from *a* in the figure instead of from *O*, we shall produce the curve corresponding to a phase difference of $\pi/8$. This means that at the instant when the *y* component passes through *AB* in the positive direction, and the *y* displacement is

therefore zero, the x component has already reached a in the positive direction, or is in advance of the y component by Oa or $\pi/8$. In like manner b corresponds to a phase difference of $\pi/4$, c to $3\pi/8$, and A to $\pi/2$. Returning toward O, it will be seen that c also corresponds to a difference of phase of $5\pi/8$, b to $3\pi/4$, a to $7\pi/8$, and O, to π, with larger values for points to the left of O.

Suppose now that we wish to trace the curve corresponding to the vibration-frequencies one to two, two for the horizontal and one for the vertical component, and with no difference of phase. Starting from O we count two points horizontally to the right and one up and reach I; again two to the right and one up for point II, and so continue, numbering the points in order until we pass through the starting point in the same direction as at first, being careful always to complete the motion in one direction before beginning the retrograde motion An excellent check upon the accuracy of the location of the points is found in the fact that points equidistant from the axis of symmetry AB, differ in number by eight in every case, that is, by half a vibration.

If now a smooth curve be traced through the points in order we see that the moving point, being subject to both motions, describes two spaces horizontally and one vertically in the same interval of time, and consequently passes through the corners of rectangles two spaces long and one space high in every case. The spaces themselves increase or decrease according to the simple harmonic law. Great diversity of figure may thus be obtained with successive differences of phase between the two component motions.

To combine two motions of frequencies two to three, we should simply count three spaces in one direction and two in the other and proceed in other respects as already described.

125. Free and Forced Vibrations. If a system when displaced from its position of equilibrium is urged to return to that position by forces, either internal or external, which vary directly as the displacement, it will, when freed from the disturbing force, execute simple harmonic vibrations about its position of equilibrium as a center. The *period* of such a vibration *depends upon the nature of the system,* and is independent of either the ampli-

tude of vibration, or of the forces tending to oppose the vibration, provided each be small. Such a vibration is called a *free vibration,* and the period is termed the *free or natural period* of the system.

In the ideal case of no friction, a body once started would continue to vibrate forever since there would be nothing to stop it. In nature, however, all vibrations are checked with more or less promptness by means of opposing forces which may be designated under the general name of friction. In such cases the amplitude gradually decreases to zero, while the period remains constant, and is independent of the friction, provided it be small. Such a vibration is termed a *damped vibration.* Examples of *free vibration* are seen in the motion of a simple pendulum, of a guitar string or of a tuning fork when bowed and allowed to swing freely.

When a system that is free to vibrate, is subjected to the action of a periodic force that varies as an harmonic function of the time, we have the conditions necessary for a *forced vibration.* The ensuing motion is the response of the system to the impressed, external force, and continues so long as the force continues. Examples of *forced* vibration are seen in the motion of the pendulum of a clock or the balance wheel of a watch, in the vibration of a tuning fork driven by an electric current, or of the sounding board of a piano or body of a violin. Since any free vibration is always more or less damped, and therefore soon sinks to zero, it follows that any *maintained* vibration is a *forced* vibration; the motion of the vibrating system being maintained by an external impressed force which varies with the time. The *period of the forced vibration* is the *period of the force,* and the amplitude is proportional to the force.

The characteristics of *free* and *forced* vibrations may be contrasted as follows: A *free* vibration gradually *dies away* on account of frictional forces. A *forced* vibration *is maintained* so long as the impressed force continues, and when it ceases the free vibration ensues and gradually sinks to zero. The *period of a forced vibration* is the *period of the impressed force,* while the period of a *free vibration* depends upon the *constitution of the system,* and is entirely independent of the forces causing it, so long as the amplitude is small.

126. Resonance. A special case of *forced vibration* is that in which the period of the impressed force coincides with the free period of the system. In such a case the system rapidly absorbs energy from the individual, periodic pulses, and soon vibrates with large amplitude. Theoretically it is due to the friction alone that the amplitude does not become infinite. *It thus appears that a system free to execute vibrations of a definite period is capable of selecting and absorbing from the surrounding medium, energy in the form of vibrations of the same period as those which it can execute.*

This is known as the *principle of resonance,* which finds its applications in every department of physics. Resonance depends upon the cumulative effect of small impulses applied to a system at exactly the proper time to produce the maximum effect.

127. Illustrations of Resonance. Two strings stretched upon a sonometer if tuned to unison, will mutually transmit vibratory motion, by means of synchronous impulses sent through the air and through their common support. If either string be set in vibration the other begins to vibrate.

Let two heavy pendulums of the same period be mounted upon a wooden frame which yields slightly to their motion, and let one be set vibrating while the other remains at rest. In a few minutes it will be seen that the second pendulum is acquiring vibratory motion through the support. Its motion gradually increases until the two are swinging with equal amplitude, but with a *phase difference of half a period.* The second pendulum continues to lag behind the first, gradually absorbing its energy until the first is brought to rest, after which the phenomenon is repeated in the reverse order.

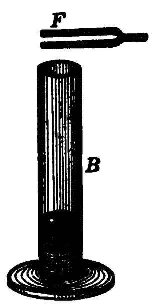


Fig. 66.

If a tuning fork be held over the mouth of a tall jar partly filled with water, it will be found on pouring in more water, that for a certain length of air column the sound of the fork is powerfully reënforced. If the fork be removed and a blast of air from a flat tube be blown across the top of the jar, the sound produced will be in unison with that emitted by the fork. The cylinder thus behaves as a stopped organ pipe and

the air column is very nearly equal to one-fourth the wave length of the sound produced by the fork. The hollow wooden cases used to support tuning forks, are in reality closed or open pipes tuned to reënforce the tone of the fork.

Let two tuning forks mounted upon suitable resonance cases and accurately tuned to the same pitch, be placed at opposite ends of a room. If one be bowed and then quieted, it will be found that the other is sounding audibly. Accurate tuning is necessary for success in this experiment. If a number of forks of different pitch be sounded together, the second fork responds to none but the one of its own frequency. Obviously the fork can absorb from the air only those wave lengths of sound which it itself can emit.

A heavy bridge is often set to vibrating vigorously by the foot-falls of a small dog trotting across it. Soldiers when crossing a bridge are commanded to break step to avoid the possibility of synchronous vibration of the bridge.

The resonators of von Helmholtz (Fig. 67), consist of hollow sphere of brass, furnished with a tubular opening for the reception of the sound wave, and opposite it a small conical tube to be inserted into the ear. The free period of the enclosed mass of air determines the pitch of the tone to which the resonator will respond. To all other tones it remains practically silent. By means of a series of such resonators von Helmholtz was enabled to pick out the various overtones in the note of a piano string, and thereby to *analyze* a sound into its constituent tones.



FIG. 67.

128. Stationary Vibrations. Suppose one end of a long flexible cord be fixed and the other end be moved quickly up and down by the hand in a vertical plane. For each up and down motion of the hand a single pulse will run the length of the cord, be reflected at the fixed end and retrace the length of the cord, to be reflected again at the hand. In each case the reflection of the pulse in the cord will involve a *change of sign* both in the motion and in the nature of the disturbance itself, since a depression in the cord is returned as an elevation, and an elevation as a de-

pression. After two reflections therefore, the disturbance will have traversed the length of the cord twice and will be identical both in *direction* and *kind* with the original disturbance, and may be regarded as starting out anew.

If now the hand be maintained in simple harmonic motion, a series of harmonic waves will run along the cord and be reflected at the fixed end as before. Each wave after two reflections will coincide both in *direction* and *kind* with the new outgoing wave, if the time required to travel twice the length of the cord be some *whole* number, *k,* times the period of the motion maintained by the hand. In other words if

$$\frac{2l}{V} = k\ T \tag{190}$$

where l is the length of the cord, V the velocity of the pulse along the cord, T the period of the motion, and k is *any interger* as 1, 2, 3, 4, etc.

Under the above condition it is clear that any disturbance however small will soon be increased sufficiently to set the cord into vibrations of wide amplitude at all points where the direct and reflected waves *coincide in phase.* Such points are called *antinodes, or ventral segments.* At certain other points, however, the incoming and outgoing waves meet in *opposite phase* and produce points of *minimum motion;* such points are called *nodes.* As a result of the superposition of the direct and reflected waves the cord is broken up into a series of vibrating segments, V, (Fig. 68), separated by points of minimum motion N.

Such a vibratory motion is called a *stationary vibration* or a *stationary wave.* The distance from one node to the next, or from one antinode to the next is *one half wave-length of the pulse in the cord.* Stationary vibrations may thus be set up in any medium capable of transmitting wave motion, and the phenomena of nodes and antinodes developed according to the principles just laid down. In all cases, the distance from *node to node,* or from *antinode to antinode* is $\lambda/2$, *for the medium in question.* From *node* to *antinode* is $\lambda/4$.

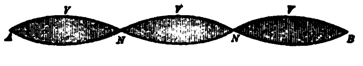


FIG. 68.

In all cases of sustained tones, as those from organ pipes, tuning forks, piano, violin, or guitar strings, the vibrating medium, whether air column, bar, or string, is executing *stationary vibrations,* and consequently presents the characteristic feature of stationary waves, *i. e., nodes* and *antinodes.*

X **129. Laws of Transverse Vibrations of Strings.** A string fastened at the ends and vibrated transversely, executes stationary vibrations as described in the previous article. The vibration of the string gives rise to a note of definite pitch, dependent upon the physical constants of the string and upon its mode of vibration. The condition for stationary vibration is

$$\frac{2l}{V} = k\,T, = \frac{k}{n} \qquad (191)$$

where n is the frequency of the note produced by the string. Of the various modes of vibration dependent upon the value of k, the simplest is that in which the string vibrates as a whole, with a node at each end. In this case $k = 1$, and

$$n = \frac{V}{2l} \qquad (192)$$

This vibration is called the *fundamental vibration,* and the tone the *fundamental* or lowest tone given by the string.

It may be shown mathematically that the velocity of a *transverse wave* in a thin, flexible string, of density d and radius r, when stretched by a tension of T dynes, is given by the expression

$$V = \sqrt{\frac{T}{\pi r^2 d}} \qquad (193)$$

If we substitute this value for the velocity in the equation for the frequency, we have

$$n = \frac{1}{2l}\sqrt{\frac{T}{\pi r^2 d}} \qquad (194)$$

Hence the freqency of the fundamental tone emitted by a string vibrating transversely, varies

(*a*) Inversely as the length of the string.

(*b*) Inversely as the radius of the string.

(*c*) Inversely as the sqare root of the density of the string.

(*d*) Directly as the square root of the tension.

These laws are deduced theoretically from the case of a long, thin and perfectly flexible string, and are very nearly realized in the case of silk or gut strings. If metallic strings are used, the rigidity of the string acts as an added tension, thus making the frequency higher than the formula would indicate.

130. Melde's Experiment. If a thin, flexible cord some four meters long be attached to some convenient source of simple harmonic motion, and the tension be properly adjusted, the laws of vibrating strings may be verified by experiment.

In Melde's experiment one end of the cord is attached to one prong of a vertical tuning fork, and the other end carrying a pan is passed over a light pulley. If the prongs of the fork stand normally to the cord, then the end of the cord will be displaced longitudinally at each vibration of the fork. At the forward swing the cord relaxes and drops down, rises and stretches tight on the backward swing of the fork, passes the position of rest and rises above on the next forward swing, falls to the middle on the backward swing and to its lowest position again on the forward swing. The fork has thus made *two complete* vibrations, for a *single vibration of the cord,* or the cord in this position vibrates an octave lower than the fork.

If the tension be properly adjusted by weights placed in the pan, the cord opens out into a wide spindle which remains fixed while the vibration contines. If the tension be reduced to one-fourth its value, the velocity of the pulse will be one-half its previous value, and the cord will now present two spindles with a node in the center. One-ninth the original tension will give three spindles and two nodes. This verifies the law of tensions.

If the fork be rotated about a vertical axis so that its vibrations are normal to the length of the cord, the cord will vibrate in unison with the fork, and for the original tension, it will divide into *two spindles,* where it previously vibrated in *one.* Since *each half* now vibrates in unison with the fork, while before the whole cord vibrated an octave *below* the fork, the law of lengths is demonstrated.

131. Segmental Vibration. It has been shown that a string may execute stationary vibrations under the conditions that

$$\frac{2\,l}{V} = k\,T = \frac{k}{n}$$

This means that the frequency of the tone emitted by a string may depend upon *its mode of vibration,* as well as upon its *tension, length, diameter or density.* In the case of a string sounding its fundamental, k is unity and the string *vibrates as a whole,* with nodes at the two ends. This is the simplest mode of vibration. The next simplest is when the string is divided into two segments with a node in the middle. In this case $k = 2$, and the frequency is double that of the fundamental. This tone may be drawn from a string by holding it at its middle point with a thin shaving of cork and bowing it lightly at about one-ninth its length from one end. Here the string may be considered as made up of two strings of equal length vibrating in unison, and producing a tone an octave above the fundamental. In a similar way the string may be broken up into three, four, five, six, or any number of equal segments, corresponding to the integral values of k. The frequency of the tones in each case is inversely as the length of the segments, and consequently if the string vibrate in three segments, the frequency is *three times* that of *the fundamental*; if it vibrate in fourths, the frequency is *four times* the fundamental; if in fifths, five times, etc.

The experimental demonstration of segmental vibration is most readily accomplished by means of a piece of piano wire about four meters long, tightly stretched between two bridges clamped to the top of a long table. One end should be attached to a key or screw in order to vary the tension at will. A series of stiff paper markers should indicate the aliquot parts of the string, as the thirds, fourths, sixths, etc. A thin shaving of cork should be slit and fixed upon the wire so as to slide freely upon it. Little riders of white and colored paper, may be distributed along the wire, with the white ones at the aliquot parts.

If the cork be held at one-eighth the length of the wire and the bow be applied, gently at first, then more vigorously, the riders at the eighths will remain seated while all the others will be thrown off. The frequency of the tone emitted by the string will

be eight times that of the fundamental. In a similar way the string may be made to vibrate in sixths, fifths, fourths, thirds and halves, the position of the cork marking a node in each case. If the cork be *not* upon some aliquot point, no satisfactory vibration and no note of definite pitch can be produced, hence *we conclude that a uniform string may vibrate as a whole, or in any number of equal parts, and the frequency of the note emitted will be proportional to the number of parts.*

132. Overtones. If the piano wire of the preceding experiment be vigorously bowed and then damped at one-fourth its length, the note will be observed to change its character. The fundamental will disappear and the second octave, a note whose frequency is four times that of the fundamental, will be heard instead. This shows that while the string was *vibrating as a whole,* it was also *vibrating in fourths,* and further, that the note emitted was made up of the *fundamental tone, and the second octave.* By successively bowing the string, and damping it at the middle, and at one-third its length, the first octave and the twelfth are found to be present when the string vibrates freely.

It thus appears that a string may at the same time vibrate as a whole and divide into several sets of equal segments, thus giving rise to the fundamental and also to tones whose frequencies are much higher than that of the fundamental. The higher tones thus obtained are termed *overtones,* or *upper partials.* In case their frequencies are exact multiples of that of the fundamental the entire series are called *harmonics,* in which the fundamental is properly termed the *first,* the first octave the *second,* the twelfth the *third harmonic,* and so on, since these tones represent a series in which the frequencies are as 1:2:3:4:5, etc.

If we consider the key of *C,* the first ten harmonics, counting the fundamental as the first, are C (1), c (2), g (3), c_1 (4), e_1 (5), g_1 (6), c_2 (8), d_2 (9), e_2 (10); where the *seventh harmonic* lies between a_1 and b_1 and may be represented by b_1 flat.

Of these tones it will be observed that the first six harmonics are consonant tones, and if sounded together, would produce the effect of a *perfect major chord.* In a string of uniform dimensions and homogeneous structure, the frequencies of the *upper partials* approach very nearly the exact relation demanded for

harmonics. It is for this reason that the music from stringed instruments is so rich and pleasing.

Again, it is clear that if the string be bowed or struck at its *middle point*, that point can not by any possibility be a node. Hence all partial tones which require the presence of a node at the middle of the string must of necessity be absent. The discordant effect of the seventh and ninth harmonics of a string are avoided in the case of a piano by having the hammer strike the wire at a distance of a little less than one-seventh the length of the string from one end.

PROBLEMS

1. Show that the right and left hand members of the equation

$$V = \sqrt{\frac{e}{d}}$$

are of the same dimensions.

2. Assuming the velocity of sound in air at 0°C. to be 331.36 meters per second, calculate the velocity in hydrogen at the same temperature, having given the mass of one liter of hydrogen = 0.0896 grams.

3. Find the temperature at which the velocity of sound in air is 356 meters per second.

4. The flash of a hunter's gun is seen and after five seconds the sound is heard. Required the distance from the observer to the hunter, the temperature being 22°C.

5. Colladon and Sturm measured the velocity of sound in the waters of Lake Geneva, and found that it travelled 1435 meters per second, the temperature being 8.1°C. Compute the coefficient of elasticity for water at this temperature.

6. A wire 50 cm. in length, and of mass 80 gm. is stretched so that it makes 100 complete vibrations per second. Compute the stretching force.

7. A string is attached to one prong of a tuning fork and

after passing over a smooth peg is stretched with a force of 32×10^3 dynes. When the string is parallel to the motion of the fork it vibrates steadily in three segments. What tension is required to make it divide into two segments? Into five segments?

8. What tension is needed to have the above mentioned string divide into eight segments, when the string stands at right angles to the motion of the fork?

9. Determine the vibration frequency of an air particle in a sound wave 10 meters long.

10. If the first syllable of an echo reaches the ear three seconds after the spoken word, how far distant is the reflecting surface? ($t = 20°C.$)

11. A stone is dropped into a well and the sound of the splash is heard after five seconds. How deep is the well, if the temperature be 10°C.?

12. An open organ pipe 120 cm. in length, is tuned correctly when the room temperature is 20°C. What will be the change in its frequency when the temperature rises to 32°C.?

13. A workman strikes a blow with a hammer upon one end of an empty iron water pipe, 600 meters long. A second workman placing one ear against the other end hears two sounds. How far apart are they in time? Temperature 20°C.

14. A string carrying a small globule of mercury is viewed through a lens fastened to one prong of a tuning fork, placed at right angles to the string. The fork has a frequency of 128. The fork is bowed and the string is tuned until the ellipse seen through the lens, makes one complete rotation in 6 seconds. If more tension be added the Lissajous' figure rotates faster. What is the per cent of error if this tuning be assumed as correct?

CHAPTER XIV

VIBRATION OF AIR IN PIPES AND CAVITIES

133. Vibration of Air Columns. In many musical instruments the vibrating body is a column of air in a pipe. Although the shape of the column and the mode of excitation may vary, yet the general principles of vibrating bodies will apply with but slight modification. When a series of similar pipes of the same diameter but of different lengths, are sounded by blowing in turn across the ends of each, it will be found that the frequencies of the sounds produced are practically inversely as the length; that is, a slender pipe 10 cm. long will give a note approximately one octave higher than a similar pipe 20 cm. long, and two octaves higher than one 40 cms. long.

If a tuning fork be held over a vertical pipe, the lower end of which is connected with a water supply for varying the length of the enclosed air column, it will be found that for a certain level of the water the air column in the pipe responds loudly to the vibrations of the fork. If pipes of different diameters are used it will be found that under similar conditions, the length of pipe responding to a given fork is nearly constant, diminishing slightly as the diameter increases. Again if a closed pipe 20 cm. long respond to a given fork it will be found that an open pipe of the same diameter and same length will respond to a fork an octave higher than the first fork. This shows that *the pitch of an open pipe is an octave higher than that of a closed pipe of the same length.*

134. Length of Organ Pipe and Wave Length of Fundamental Tone. (*a*) *Open pipe.* Suppose an open pipe, Fig. 69, have placed before one end, a tuning fork or other suitable vibrator, which sends a series of sound waves into the pipe. Then *for the pipe and the tuning fork to vibrate in unison,* it is necessary for the reflected wave to return to the end *B,* in the

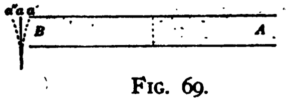


FIG. 69.

proper phase to unite with the outgoing wave. This means, that when the fork starts to swing from a'' to a' a condensation is sent into the pipe and runs the length BA while the prong moves the distance $a''a'$. At the open end A, the condensation is reflected as a rarefaction (Article 117), which starts into the tube at A at the same instant that a rarefaction enters at B, due to the backward motion of the prong from a'. These two rarefactions meet at the center of the tube, producing a *double rarefaction* for an instant, and pass on to the open ends to be reflected as condensations, at the instant the fork begins a second swing from left to right. The condensation at B unites with a new condensation from the fork, and the combined condensations run in at B while the reflected condensation enters at A. The two condensations meet at the middle forming a *double condensation* at that point, just half a period later than the double rarefaction.

The disturbance is thus seen to travel the length of the pipe twice during one complete vibration of the fork, or, for the fundamental tone of the open pipe

$$\frac{2l}{V} = T \tag{195}$$

whence

$$2l = VT = \lambda \tag{196}$$

This shows that for an open organ pipe *the wave length of the fundamental is twice the length of the pipe.*

(*b*) *Closed pipe.* In the closed pipe the fork on its swing from left to right sends in a condensation which runs to the closed end and being reflected as a condensation runs back to the open end where it emerges and combines with the out-going condensation caused by the fork on its swing from right to left. At the same time the emerging condensation is reflected *into* the pipe at the *open end* as a rarefaction which combines with the rarefaction left in the rear of the fork on its passage from right to left. The condensation or the rarefaction has in each case run the length of the pipe *twice* during *half* a vibration of the fork, or for unison with the fundamental,

$$\frac{4l}{V} = T \tag{197}$$

whence

$$4l = VT = \lambda \tag{198}$$

That is, the wave length of the fundamental of a closed organ pipe is *four times the length of the pipe.* If we compare this result with that obtained for the open pipe, we see that the wave length of the closed pipe is double that of an open pipe of the same length, or the fundamental of a closed pipe is an octave lower than that of an open pipe the same length.

135. Nodes in Open and Closed Organ Pipes. In the open pipe it was shown that a node existed in the middle, at which point there existed alternately double rarefactions and double condensations at intervals of half a period. A node therefore in a vibrating air column is to be considered *as a place of maximum change of density, but of minimum motion.* In an open pipe there is *always an antinode at each end,* since at the open end the motion is unrestricted. For the fundamental in an open pipe therefore, there is a node in the middle and an antinode at each end, or the pipe contains *one-half wave length* (Fig. 70-*A*).

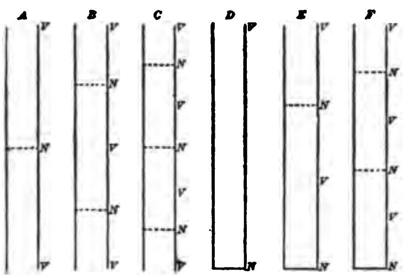


FIG. 70.

If the pipe be blown more strongly it gives its first octave, the air column breaks up into segments, with a node at one quarter the length of the pipe from each end, and an antinode at the middle and at each end. In this case the pipe contains *two half wave lengths,* and the corresponding note is an octave above the fundamental (Fig. 70—*B*).

The second overtone is given by the air column forming three nodes, one at *one-sixth* the length of the pipe from either end and at the middle, with antinodes between. In this case the pipe contains *three half wave lengths,* and the frequency of the tone is three times that of the fundamental (Fig. 70—*C*), and so on for higher tones. *Hence in an open pipe all the overtones are present.*

In a closed pipe, *there is always a node at the closed end,* since the air is at rest there, and as usual there is an antinode at

the open end. For the fundamental the pipe contains ***one-fourth wave length*** (Fig. 70—*D*).

The first overtone in the closed pipe is given by the air column forming a new node, at one-third the distance from the open end. The pipe thus contains *three fourth wave lengths,* and the tone has three times the frequency of the fundamental (Fig. 70—*E*). The second overtone produces a node at *one-fifth* and *three-fifths* the length of the pipe from the open end. The pipe contains *five fourth wave lengths,* and the frequency of the tone is *five* times that of the fundamental (Fig. 70—*F*). *Hence in closed pipes only those overtones are present whose vibration frequencies correspond to the odd multiples of the fundamental.*

136. Quality of Sound. By means of his analysis of musical sound von Helmholtz decided that the *quality* of a sound depends upon the number of overtones associated with the fundamental and upon their relative intensities, and is independent of their differences in phase. Quality of sound depends upon the form of the sound wave. In the ear the various constituents of a complex wave are separated and noted, and the effects of the various combinations distinguished. Von Hemlholtz showed not only by direct analysis, but also by synthesis that the sounds of certain musical instruments consist of definite overtones combined with the fundamental. By means of a series of tuning forks each of which gave a simple tone, he was able successfully to reproduce the notes of various musical instruments, and even to imitate most of the vowel sounds of the human voice.

An admirable instrument for the analysis of sound is found in the manometric capsule devised by Koenig. A cylindrical box (Fig. 71), is divided into two separate, air-tight compartments by a flexible diaphragm *D*, of thin rubber, or gold beater's skin. Into one of the compartments *A*, are introduced the sound waves by means of the funnel *M*. The compartment *B*, contains illuminating gas, which enters through the tube *C*, and is ignited

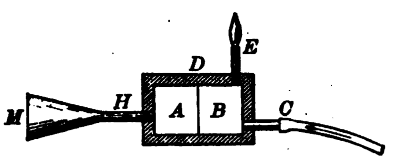


FIG. 71.

at the tip E. If a condensation impinge upon the diaphragm, the gas in B, is compressed and the flame leaps up; if a rarefaction enter A, the pressure in B is less and the flame is drawn down. If a musical note be sung into the funnel, the flame vibrates in unison with the air particles in A, and if it be viewed in a rotating mirror, the eye can determine at once the nature and constitution of the sound. In Fig. 72 the upper picture represents the appearance in the mirror when a simple tone is sounded in the funnel. The middle line represents the appearance of the flame when the octave of the first note is sounded, and the third shows the effect of combining the two. Manometric capsules may be attached to each of a series of resonators and the combination affords a means of instantly determining the composition of any note sounded in its vicinity.

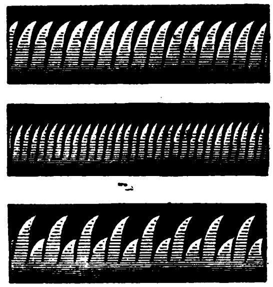


FIG. 72.

137. Kundt's Experiment. The principle of resonance has been ingeniously applied by Kundt to the measurement of the velocity of sound in solids or in gases. A long glass tube

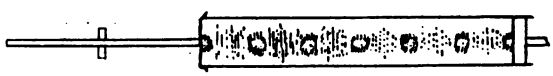


FIG. 73.

(Fig. 73), some 6 cm. in diameter is furnished at one end with a loosely fitting piston, and has the other end closed by a sheet of thin rubber. A rod of brass or other metal is held by its middle point in a vise and one end is furnished with a disk of stiff paper which rests against the rubber membrane of the glass tube. The inside of the tube is dusted with fine cork filings or amorphous silica.

When the rod is stroked with a piece of chamois skin covered with a little rosin, it gives a loud clear note. It vibrates lon-

gitudinally, with a node in the center, after the manner of the air in an open organ pipe. The paper disk communicates the vibrations of the rod to the air in the tube, and when the length of the enclosed air column has been properly adjusted by means of the piston, the cork dust is tossed about into little heaps owing to the resonance of the air with the note emitted by the rod.

As we have already seen, the conditions of resonance demand that the path run over by the disturbance from the end of the rod, to the end of the air column and back again, must be some even number of wave lengths, since the tube is closed at each end, and the motion is twice reflected with change of sign in direction, but without change of sign in condensation. When resonance has been established the powder in the tube assumes the appearance shown in Fig. 73. The nodes being points of minimum motion, are marked by small circles of the cork filings, while the antinodes are shown by transverse heaps of dust, where it has fallen at the cessation of the sound.

In Article 125, it was shown that the distance from *node* to *node* is a half wave length of the disturbance in the medium. The nodal lengths in the air may be readily measured from the circles of powder, and the half wave length of the sound in air computed. Since the rod behaves as an open organ pipe sounding its fundamental, it follows that the length of the rod is *one half wave length of the sound in brass.* If V_a and V_b, λ_a and λ_b represent the velocities and wave lengths of sound in air and brass respectively, then, since the period is the same in each case, we have

$$\frac{V_b}{V_a} = \frac{\frac{\lambda_b}{2}}{\frac{\lambda_a}{2}} \qquad (199)$$

or

$$V_b = V_a \frac{\lambda_b}{\lambda_a} \qquad (200)$$

For gases, an additional tube with powder is fitted up and placed in contact with the other end of the rod. From the measured

lengths of the nodal distances in the gas and air the computation is made as given above.

138. Mouth Pieces. The various forms of wind instruments differ chiefly in the mode of excitation of vibration of the enclosed column of air. That part of the instrument in which such vibration is excited is called the mouth piece. Mouth pieces may be divided into three classes.

(*a*) Those in which the air is blown *across* a sharp edge or across an opening, as in the common tin whistle, the organ pipe, or the flute.

(*b*) Those in which the air is forced *through* an opening, either partially closed by an elastic tongue or reed which swings through, as in the common cabinet organ, harmonica, accordion, etc., or closed by a reed which shuts *down upon* the opening, as in the clarionet, oboe and bassoon.

(*c*) Those in which the air is forced through a slit formed of two elastic membranes. This form of mouth piece is made by cutting off the ends of a wooden tube obliquely on opposite sides and tieing two strips of thin rubber over the faces so formed, so as to leave a narrow slit along their line of junction. If air be blown through the slit a note will be produced, whose pitch will be modified by the body of air in the tube.

139. Vocal Organs. Of all musical instruments the larynx, the organ of human speech and song, is the most wonderful, both on account of its simplicity as well as for its extreme delicacy and range.

The larynx may be briefly described as box formed of three plates of articulated cartilage which are moved by muscles, placed at the upper end of the trachea or windpipe.

The base of the larynx is formed of a large ring of cartilage called the *cricoid* (ring shaped) cartilage. Attached to this is the *thyroid* or shield-shaped cartilage, which is bent in the shape of a *V*, and fastened to the edges of the *cricoid* ring by its sides, the point of the *V* being turned to the front, forming the projection on the front of the throat known as "Adam's apple." At the back of the *cricoid* are fastened two small pointed cartilages, the *arytenoid* (funnel shaped) cartilages. Stretching between the *arytenoid* cartilages to the inner sides of the *V* shaped *thyroid*

are two elastic membranes, one fastened to each leg of the *V*. These are the *vocal chords, cc,* Fig. 74. Just above these are two folds of mucous membrane, known as the *false vocal chords, ff,* Fig. 74.



Fig. 74.

When the peak of the *thyroid* is not drawn down, the vocal chords are lax and the breath passes freely between them as in ordinary breathing. But when the peak of the thyroid is pulled down by the muscles attached to it, the vocal chords are stretched, the arytenoid cartilages move nearer to each other, and the thin, sharp edges of the vocal chords form a narrow slit across the wind pipe, through which the air is forced causing them to vibrate as the rubber membranes in the third form of mouth piece described in Art. 138.

The pitch of the tone produced by the vocal chords, depends upon their size, length and tension. The tension is controlled by the attached muscles, so that the singer can vary at will the pitch of the tone produced.

The quality of the sound produced, is modified by the resonant qualities of the cavities of the mouth, throat, and nasal passages.

140. The Ear. The human ear consists of three well marked divisions, termed the *external, middle* and *internal ear.* The external part includes the familiar appendage at the side of the head, and the auditory canal, or *meatus, M* (Fig. 75), in which is shown a section through the right ear.

The meatus is closed by the *tympanum,* or eardrum, *m t.* This separates the external from the middle ear, and acts as a receiving membrane against which the sound waves impinge.

In the middle ear is found a chain of three small bones—the hammer (*malleus*), *m*; the anvil (*incus*), *i*; and the stirrup (*stapes*), *s.*

The handle of the hammer is attached to the tympanum and the base of the stirrup rests against the membrane of the *oval window,* a small oval opening into the inner ear. The middle

ear communicates with the external air by means of the *Eustachian tube E,* which leads to the upper part of the throat.

The internal ear or labyrinth, is seated deep in the skull, and communicates with the middle ear by two openings in its bony case, the *oval window,* and the *round window, fr.* It consists of three parts: the *vestibule, V*; the *semicircular canals, h, vp, va*; and the *cochlea,* or snail-shell, *c,* all of which are filled with a watery fluid.

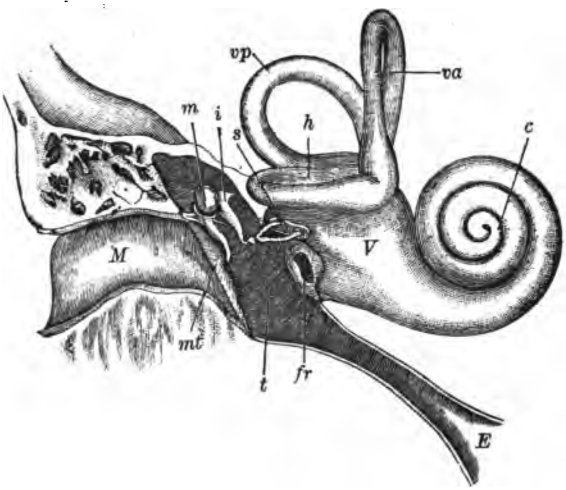


FIG. 75.

The cochlea is a tapering tube, coiled up like a snail-shell, and divided longitudinally, into two compartments. These are formed by a bony partition, extending out from the axis of the spiral, and two membranes joined to its edge and attached to the walls of the tube. In the fluid of the cochlea between the two membranes, are found about 3000 rods of different length, known as the rods of Corti. These rods are fastened to the lower membrane, standing in two parallel rows, which extend its whole length. The rows lean toward each other, each rod being joined to its opposite neighbor. A nerve filament from the brain is connected to each pair of rods. A large number of stiff, elastic hairs are also found floating in the fluid of the vestibule and attached to the membrane on its sides.

The process of hearing begins in the transmission of the sound waves to the drum of the ear through the air. From the ear-drum the vibratory motion is transferred by the small bones, to the yielding membrane of the oval window, and transmitted to the enclosed liquid of the vestibule and the cochlea, by which it is transmitted to the nerve filaments attached to the rods of Corti. It is thought that the function of these rods is to resolve sounds into their components and to report the individual components to the brain. It is assumed that such resolution is effected through resonance; each pair of rods responding to some specific tone. The hair like extensions of nerve filaments in the vestibule are thought to aid in the perception of sounds that are lacking in tonal or musical quality.

The semicircular canals *h, va,* and *vp* (Fig. 75), are so placed that one lies in a horizontal plane, the second in a vertical plane from front to back, and the third in a vertical plane from right to left. These canals seem to have no connection with the sense of sound, but are regarded as the organ of the sense of equilibrium. Animals from which one or more of these tubes have been removed seem to experience great difficulty in maintaining their equilibrium.

OPTICS

CHAPTER XV

FUNDAMENTAL PHENOMENA

141. Definitions. Optics is that branch of physics which has for its object the study of the nature of light and the circumstances of its propagation. In *Geometrical Optics,* the circumstances of the transmission of light are deduced from certain laws established by experiment. In *Physical Optics* these laws are explained, in accordance with certain assumptions regarding the constitution of matter and the nature of the space which it occupies. Geometrical optics makes no assumptions regarding the *nature* of light, but through the application of certain geometrical laws which are true whatever the nature of light may be, it deduces important formulæ and explains many optical phenomena. Physical optics, on the other hand, explains many phenomena that cannot be accounted for upon geometrical principles, and anticipates in many cases the results of experiment.

A body which emits light of itself as the sun, the fixed stars, lamps, etc., is said to be *self luminous.* Bodies which emit no light of themselves, but which shine by reflected light are called *non-luminous* bodies.

If a body transmit light freely, it is said to be *transparent.* If it transmit light but poorly, so that the outlines of a body can not be seen through it, it is called *translucent.* An *opaque* body is one that transmits no light. As usual in physical properties these distinctions are not absolute. No substance is perfectly transparent, neither is any entirely opaque.

A *ray* of light denotes nothing more than the straight line along which light is propagated. A symmetrical collection of rays about some axis is called a *pencil.* If the rays meet at a point, such a point is called the *focus* of the pencil.

Any space or substance in which light can be propagated, is called a ***medium.*** Only istropic media are considered in geometrical optics. In the study of the transmission of light in different media, the plane in which the ray is assumed to lie, is the plane containing the normal to the surface of the medium. The angle of *incidence,* of *reflection* or of *refraction* is the angle included between the normal and the *incident, reflected* or *refracted* ray. In isotropic media, these angles all lie in the same plane.

142. Nature of Light. The most casual observation shows that when light is transmitted from one body to another, energy is transmitted at the same time. Living plants and animals are stimulated and influenced by the action of light. Certain chemical reactions are induced by light; a mixture of hydrogen and chlorine in equal volumes explodes on exposure to the sunlight.

Again, light is developed only at the expense of energy in some other system, as in the falling stars and meteors, in the filament of an incandescent lamp, in the flame of a candle or in the faint glow of the fire-fly. The source of all energy is the sun. The energy of the sun requires a little over eight minutes to traverse the gulf of intervening space between the sun and the earth. The question arises, where was the energy during that interval of eight minutes? Now since the only way in which energy can be transferred from one body to another remote from it, is either by the bodily transference or matter, as in the case of a projectile, or by disturbing the medium surrounding the second body, the answer to the foregoing question necessitates a choice between two assumptions. Either we are to assume that light is transmitted across space in the form of small particles which carry the energy with them, or we must assume the existence of a continuous medium in space through which light is transmitted in some form of wave motion. No third mode of transferring energy is conceivable.

The first of these assumptions lay at the bottom of the corpuscular or emission theory as elaborated by Newton. The assumption of an hypothetical medium underlies the undulatory theory as founded by Huyghens and developed by Young and Fresnel. This hypothetical medium is called the ether. It is assumed to fill all space and permeate all bodies, to be incom-

pressible, of infinite elasticity, to offer no resistance to the passage of bodies through it, yet to be capable of resisting a shearing stress. Such a medium is at present absolutely necessary for the explanation of the phenomena of light, heat, and electricity.

The undulatory theory of the propagation of light, is universally accepted today, and although in the articles under Geometrical Optics no theory of propagation is mentioned, since none is needed, it is to be understood as being tacitly assumed. While there is nothing in the geometrical treatment inconsistent with the undulatory theory, yet regard for simplicity has suggested that its formal statement be deferred until a later chapter.

143. Rectilinear Propagation. One of the earliest facts of observation upon optics is that light in a homogenous medium, appears to travel in straight lines. Opaque bodies are illuminated only on the side turned toward the source of light. It is impossible to see through a tube bent so that no straight line can be passed through it. The mechanic, the surveyor and the astronomer alike recognize and use this fundamental fact.

A closer examination of the phenomena will show, however, that rectilinear propagation of light is not only confined to homogeneous media, but is only approximately true even there. It will be shown later that owing to its fundamental nature, light does bend round corners, and the edges of shadows of small objects are not of the size and sharpness demanded by rectilinear propagation.

In accordance with the principle of Huyghens, the transmission of light in straight lines is a direct consequence of the interference of the subsidiary waves, which start out from every point upon the wave front. So long as the physical properties of the medium are the same in all directions, the wave fronts are spherical in form and transmission of the luminous disturbance takes place in right lines normal to the wave front. In media of varying density, or of unequal elasticity, the wave form is no longer spherical, and the velocity of the disturbance is different in different directions. In media in which the density varies gradually from point to point, the transmission of light may even take place along curved lines. This is exemplified in the distortion of images of objects seen through air rising from heated surfaces, in the effects of mirage, and in the twinkling of stars.

144. Shadows. A result of the rectilinear propagation of light is the formation of dark spaces in the rear of opaque objects of any size when exposed to a source of light. The dark outlines of an opaque body projected upon a screen by a luminous point is called its shadow. Such an outline (Fig. 76), may be found by

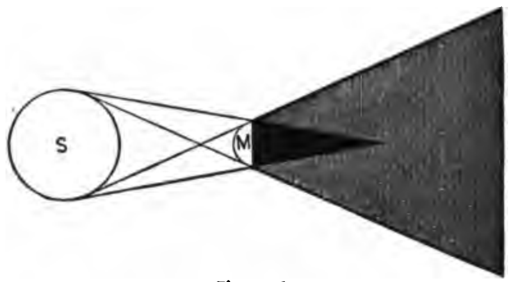


FIG. 76.

drawing straight lines from the luminous point to the screen past every point of the exposed surface of the body. The figure obtained is termed the *geometrical shadow.* If the source of light be very nearly a point, then all the light is cut off from the screen, over a certain space, which is called the *umbra.* If the luminous body have any dimensions there will always be a ring of *partial shadow* surrounding the umbra. This is called the *penumbra.*

If the luminous body be larger than the opaque body the umbra has a definite length. In the case of spheres the umbra becomes a cone of shade extending out into space, surrounded by a region of penumbra which increases with the distance from the opaque body. In eclipses of the moon, the moon passes through the cone of earth shadow, entering the penumbra first and leaving it last. In solar eclipses the tip of the cone of shadow from the moon may or may not extend to the earth, according to the position of the moon in its orbit, since the mean length of the lunar shadow is less than the mean distance from the moon to the earth. If the shadow cone reach the earth, the solar eclipse is *total* for all points touched by the umbra; for points touched by the penumbra only, the eclipse is *partial.* If the tip of the cone fall short of the earth, the eclipse is *annular* for the successive

points touched by the prolongation of the axis of the cone of shadow.

145. Images Through Small Apertures. An *image* may be defined as the picture of an object, formed by the crossing, either real or apparent, of a series of rays from the various points of an object. If the rays actually pass through the image it is a *real image,* and may be caught upon the hand or projected upon a screen. If the rays *seem* to come through the image but do not, the image is *virtual.* If a luminous body *CD,* (Fig. 77), be placed in front of a small aperture in an opaque screen *A,* there will be depicted upon a second screen *B,* an inverted image of the body. This arises from the rectilinear propagation of light. Each point on the luminous body acts as an independent source of light, from which a cone of rays pass out toward the screen.

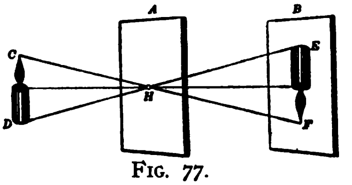


FIG. 77.

Since the only rays from the extremities of the object that can reach the screen *B,* are *CH* and *DH,* it follows that all the light reaching the screen from the point *C,* will be collected at *F* on the image, and similarly for the point *D* and its image at *E.* The position of the image is consequently *inverted,* since all the rays forming the image cross at the aperture. The image, if viewed from the side toward the aperture is also *perverted,* i. e., the parts of the image relative to the object are symmetrically displaced with regard to a vertical plane through its center and normal to *B.* A glove for the right hand is the perverted image of that for the left hand. No amount of rotation can make them agree. An image seen in a plane mirror is perverted.

The *distinctness* of the image has reference to the sharpness of its outline. If the aperture *H,* be made smaller, fewer rays from the point *C,* would reach *F,* but these would all be from *C,* hence there would be no overlapping of images. The *distinctness,* therefore, varies *inversely* as *the size of the aperture.*

The *brightness* of the image refers to the amount of light per unit area, that reaches the screen *B.* Manifestly the brightness will be greater the larger the aperture, or the brightness *varies directly as the size of the aperture.*

The *size* of the image may be determined from geometrical considerations. The two triangles *CHD* and *EHF* are similar, and we see at once that the *size of the image is to the size of the object, directly as their distances from the aperture.*

It remains to be noted that the conditions for *brightness* and *distinctness* are mutually opposed, and that with a simple aperture one must be sacrificed to obtain the other. A converging lens placed in the aperture will secure both *brightness* and *distinctness* at the same time.

GEOMETRICAL OPTICS

CHAPTER XVI

PHENOMENA OF REFLECTION

146. Reflection of Light. When light, travelling in one homogeneous medium, meets the boundary of another homogeneous medium of different optical density, it is in general divided into several parts which follow different paths. A part is *reflected,* or turned back into the first medium along definite lines. A part is *scattered,* or reflected in all directions, a part is *transmitted* along one or two new paths, and a part is *absorbed.*

Scattering of light is produced by reflection at irregular or rough surfaces. All non-luminous bodies are rendered visible by scattered or diffused light. Even the most highly polished mirrors scatter some light. In general the quantity of light reflected from a surface increases as the angle of incidence increases. In one case to be discussed later, the reflection is *total.*

When a beam of light is allowed to strike the surface of a piece of highly polished glass or metal, the light leaves the surface along definite lines and is said to be reflected, as in Fig. 78, which shows the reflection of light by a plane mirror. The angle *IAP* is the *angle of incidence,* and the angle *PAR* is the *angle of reflection.* The law of reflection of light states that *the angle of incidence is equal to the angle of reflection, and the two angles lie in the same plane.*

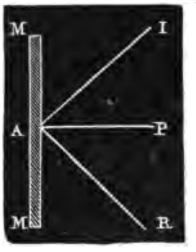


Fig. 78.

The experimental proof of this law lies in the fact that with the best mirrors and circles that men have been able to make, no one has ever found the slightest deviation from its truth.

147. Images in a Plane Mirror. When an illuminated object is placed before a plane mirror, rays of light pass off from

it to the mirror and are reflected to the eye. An image of the object is seen reflected in the mirror, in the direction *from which the light enters the eye.*

In accordance with the law of reflection, any two rays, AB and AC, (Fig. 79), will, if produced backward after reflection, seem to meet along the lines DB and EC, as if they originally came from A', and an eye so placed as to receive these two rays would seem to see the point A, at A'.

The position of A' is readily found. From A drop a normal to the mirror cutting it in K, and extend it to meet BD produced backward. Then the angles

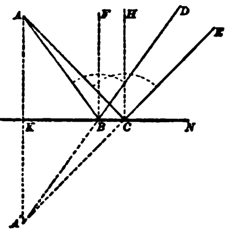


FIG. 79.

$$ABM = DBN = KBA',$$

and the angles at K are right angles, hence the triangles AKB and $A'KB$ having the side KB common and the three angles equal each to each, are similar and equal. Therefore $AK = KA'$, or *the image seen in a plane mirror lies as far behind the mirror as the object lies in front of it.*

Again it may be shown that the triangles AKC and $A'KC$ are equal, whence the triangles ABC and $A'BC$ are equal in all their parts, or the angles BAC and $BA'C$ are equal. This shows that the divergence of the rays before and after reflection is equal and hence the image is not distorted.

148. Path of Rays. The image of an object seen in a plane mirror may be constructed, and the paths of the individual rays determined graphically, in accordance with the foregoing principles. Let AB (Fig. 80), represent an object in front of the plane mirror MN. If the positions of the images of A and B can be determined, the complete image may be obtained by joining these two points. From A and B drop perpendiculars upon the mirror and produce them as far behind the mirror as the points A and B lie in front of it. The positions of the points A' and B' are thus determined.

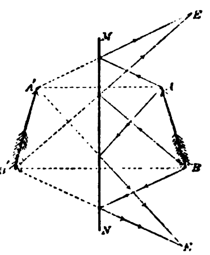


FIG. 80.

To an eye situated at E or E' rays seeming to come from A', really touch the mirror at the points where a straight line from E or E' to A' intersects the mirror, and since the rays originally came from A, these points are to be joined to A. The paths of the rays from B are found in the same way.

149. Deviation Produced by Rotation of Plane Mirror. If a ray of light fall upon a plane mirror, and the mirror be turned through any angle, the reflected ray will be turned through twice that angle. For let AM (Fig. 81), be a ray of light incident normally upon the mirror M. The ray after reflection retraces its path, since the angles of incidence and reflection are both zero. When the mirror is turned through an angle θ, the normal to the mirror is rotated through the same angle. The angles of incidence and reflection are now each equal to θ, while the total deviation of the ray is 2θ. while the total deviation of the ray is 2θ.

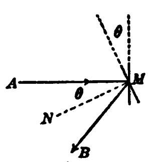


FIG. 81.

This principle finds wide application in physical apparatus where it is desired to detect small angular movements or measure the same with great accuracy. Examples of such applications are seen in the rotating mirror as applied to the examination of manometric flames, and to the measurement of the velocity of light; in the reflecting galvanometer, where a small mirror attached to the moving system indicates very slight angular displacements, and in the optical lever, where it is applied to the measurement of small lengths.

150. Successive Reflection from two Mirrors. Let AO and BO (Fig. 82), represent two plane mirrors inclined to each other at an angle ϕ, and let $PQRST$ . . . be a ray of light which is reflected in turn at Q, R, S, T . . . Let $\theta_1, \theta_2, \theta_3$ . . . be the acute angles included between the ray and the mirror surfaces at the successive reflections. Then from the triangle QOR, we have

$$\theta_2 = \theta_1 + \phi$$

and from RSO we get (201)

$$\theta_3 = \theta_2 + \phi$$

and so on.

These equations may be written

$$\theta_2 - \theta_1 = \phi$$
$$\theta_3 - \theta_2 = \phi$$
$$\cdots\cdots\cdots \qquad (202)$$
$$\theta_{n+1} - \theta_n = \phi$$

whence by addition

$$\theta_{n+1} - \theta_1 = n\,\phi \qquad (203)$$

When n is even, the angles θ_{n+1} and θ_1 are measured from the same mirror and the difference is the angle between the initial and final directions of the ray. Hence in this case *the total deviation is equal to n times the angle included between the two mirrors.*

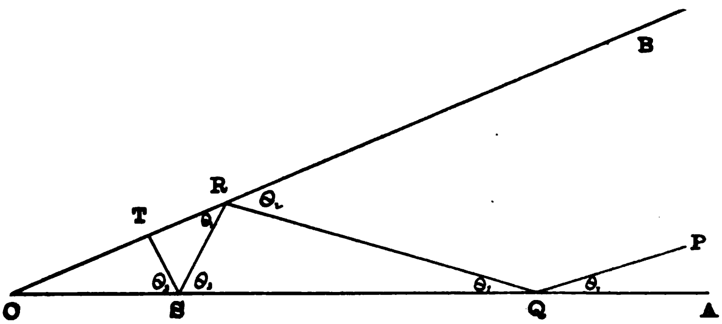


FIG. 82.

Again, the deviation is the same whatever be the angle of incidence, so that the divergence of any two rays is not changed by successive reflections.

When the ray is twice reflected, once at each mirror, the total deviation is twice the angle between the two mirrors. This principle is applied in the sextant, an instrument invented by Newton and constructed by Hadley for measuring the angular distance between two remote objects. By means of the sextant the mariner is able to determine his latitude and longitude at any time when the sun is visible.

151. Concave Spherical Mirrors. A concave spherical mirror (Fig. 83), is a part of a spherical shell with its inner sur-

face highly polished. The *center of curvature O,* of the mirror is the center of the spherical shell of which the mirror is a part. The *vertex A,* of the mirror is the center of figure of the mirror. A right line AU, joining the center of curvature and the vertex is called the *principal axis* of the mirror. From geometry it is evident that any line from O will cut the mirror normally, or a ray of light passing through the center of curvature will strike the mirror at zero incidence and be returned along the same path, and any ray striking the mirror at any point will form equal angles of incidence and reflection with the radius at that point.

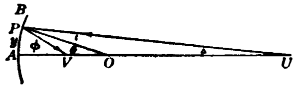


FIG. 83.

Let a luminous point be placed at U, (Fig. 83), upon the principal axis UA. Consider two rays UA and UP. Let the first pass through the center of curvature O. After reflection it will return upon its path. Let the second be reflected at any point P, making equal angles with the normal OP, and crossing the first ray at V. This point is then the image of the point U, since UP is *any* ray from U meeting the mirror. All rays, therefore, from U, will meet after reflection at V, and conversely, all rays from V, will meet after reflection at U. Two points so related are called *conjugate points or conjugate foci.* The point V is a *real focus* since the rays actually pass through it after reflection.

It is required to express the relation between the conjugate focal distances AU and AV, and the radius of the mirror. Let the acute angles at U, O and V, be denoted by Δ θ and ϕ, respectively, and let i denote the angle of incidence and the angle of reflection at P.

Then from the triangles OPU, and VPO we have

$$\theta = \Delta + i \tag{204}$$

$$\phi = \theta + i \tag{205}$$

whence

$$\phi - \theta = \theta - \Delta \tag{206}$$

or

$$\Delta + \phi = 2\theta \tag{207}$$

Let $UA = p$, $VA = p'$, and $OA = r$. If P be taken near the point A so that the angles ϕ, θ and Δ are very small, and hence may be set equal to their tangents, we have

$$\frac{AP}{p} + \frac{AP}{p'} = \frac{2AP}{r} \tag{208}$$

or

$$\frac{1}{p} + \frac{1}{p'} = \frac{2}{r} \tag{209}$$

a fundamental formula.

In the convex mirror the image is virtual and appears to be behind the mirror. By considering the quantity p negative the above formula may be shown to hold for the convex mirror as well.

152. Discussion of Formula. The formula given in (209) is based upon the assumption that the angular opening of the luminous pencils shall be *small,* that is, that the point P shall be near the principal axis of the mirror. Within these limits it is true for any value of p and hence for p equal to infinity. In this case

$$p' = \frac{r}{2} \tag{210}$$

This means that for an infinitely distant source of light, *i. e.*, for parallel rays, the *focus lies half way between the center of curvature and the mirror.* This focus for rays parallel to the principal axis is called the *principal focus.*

Again, since the sum of the reciprocals of p and p' is a constant, we see that as p *increases,* p' *decreases,* and *vice versa.* This means that *image and object move in opposite directions at all times.* Suppose p to decrease from plus infinity to r, that is, suppose the object to move from infinity to the point O, where $p = r$. We find from the formula that p' is also equal to r. Or image and object meet at the center of curvature. As p continues to decrease, the image moves away toward infinity, or p' becomes infinite for $p = r/2$. This means that for values of p less than

$r/2$, the image is *virtual and lies behind the mirror*. In other words, as the object passes the principal focus moving toward the mirror, the image shifts frem plus infinity to minus infinity and moves u ptoward the mirror from behind, and object and image meet again at the mirror.

153. Construction of Images in Concave Mirror. Having given a concave mirror *MN*, (Fig. 84), with center of curva- at *C*, and principal focus at *F*, it is required to construct the image of an object *AB*, placed beyond the center of curvature. Since the intersection of two lines is sufficient to locate a point, it will suffice to trace the path of *two rays* from any point on the object in order to locate the corresponding point on the image. Obviously *any two* rays may be chosen for the purpose, since the angles of incidence and reflection may be made equal by construction. It is convenient, however, so to choose the rays from each point that one ray shall lie parallel to the principal axis, and the other shall pass through the center of curvature, since in these two cases the direction of the rays after reflection is perfectly determined, as the first ray will pass through the principal focus and the second will retrace its path.

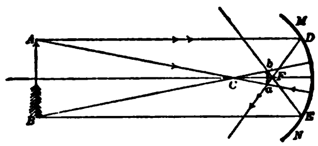


FIG. 84.

Constructing two rays in this manner from the point *A*, it is found that they cross *after reflection* at the point *a*, which is, therefore, the real image of *A*. In the same way the image of *B* is found to lie at *b*, and the image of the object may be sketched in between these two points.

The image seen in a concave mirror, of an object placed *beyond the center of curvature*, is found to be *real, inverted, smaller than the object, and located between the center of curvature and the principal focus.* Also, since rays from *a* and *b* would, after reflection, meet *A* and *B* respectively, it is clear that *a b* may be considered as an object and *AB* the corresponding image.

In case the object lie between the mirror (Fig. 85), and the principal focus, the reflected rays are *divergent,* and will not meet in front of the mirror. If, however, the reflected rays be produced **backward,** they seem to meet in the virtual points ***a*** and ***b*,** and the image may be sketched in as shown in the figure. In this case it is seen that the image of an object placed between the principal focus and the mirror is *virtual, erect, larger than the object, and situated behind the mirror.*

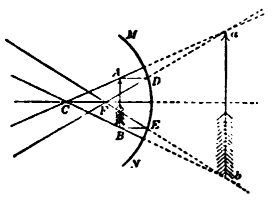


FIG. 85.

In the case of convex mirrors (Fig. 86), the center of curvature and the principal focus are both behind the mirror. Otherwise the construction is the same as in the concave mirror.

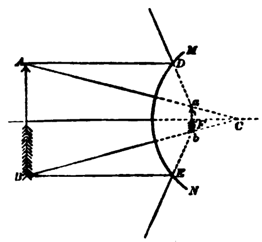


FIG. 86.

Again, if we denote by S and s the points of intersection of the principal axis with the object $A\,B$, and the image $a\,b$, in Figures 84, 85, and 86 we have a pair of similar triangles, ACB and aCb, in each figure from which we may write down at once the proportion

$$\frac{a\,b}{A\,B} = \frac{C\,s}{C\,S} \qquad (211)$$

which shows that in all cases *the size of the image is to the size of the object directly as their distances from the center of curvature.*

CHAPTER XVII

PHENOMENA OF REFRACTION

154. Refraction. When a ray of light passes obliquely from one medium into another of different optical density, it undergoes a change in direction at the surface of separation of the two media. The portion entering the second medium is said to be *refracted.*

If MN (Fig. 87), represent the surface of separation between the two media, where the lower medium is the one of greater optical density, then BA is the *incident* and CA the *refracted* ray. The plane BAD containing the incident ray and the normal AD, at the point of incidence is the plane of incidence. The plane CAE is the plane of refraction. The angle DAB is the angle of *incidence,* CAE the angle of *refraction,* and HAC the angle of *deviation.*

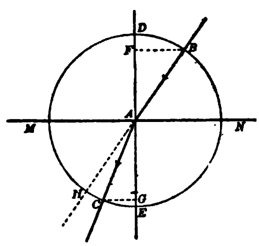


FIG. 87.

The law of refraction, first discovered by Snell in 1621, states that *the incident and refracted rays lie in the same plane, and that the ratio between the sine of the angle of incidence and the sine of the angle of refraction is a constant, depending only upon the nature of the two media.*

Mathematically the law may be stated

$$\frac{\sin i}{\sin r} = \mu = \text{a constant} \qquad (212)$$

where i denotes the angle of incidence, and r the angle of refraction.

The quantity μ is termed the *index of refraction* for the two media in question. In case the light passes from vacuum the ratio is called the *absolute index* of refraction. When it passes from one medium to another it is termed the *relative index.* Again, if the ray BA (Fig. 87), be conceived as passing from air to water then

$$\frac{\sin DAB}{\sin r} = \mu_{aw} \qquad (213)$$

If, however, the ray be reversed it will retrace its path from C to B and we have

$$\frac{\sin CAE}{\sin DAB} = \mu_{aw} = \frac{1}{\mu_{wa}} \tag{214}$$

or the *relative index from water to air, is the reciprocal of the index from air to water.* The value of μ_{aw} is 4/3, while μ_{wa} is 3/4. Usually the index is given for the light passing from the rarer to the denser medium.

155. Construction for Refracted Ray. Given two media as air and water and a ray of light BA, incident upon their surface of separation at an angle BAF (Fig. 88), in the rarer medium, it is required to construct the refracted ray. About the point of incidence A draw two concentric circles with radii AD and AF whose ratio is 4/3, the index of refraction for the two media involved. Continue the line BA until it cuts the inner circle in E. Draw EC parallel to the normal DA and drop the perpendiculars BF, EH, and CG upon DA. Finally, draw AC which is the path of the refracted ray.

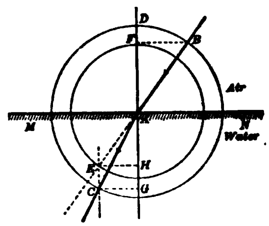


FIG. 88.

The angle of incidence $BAF = EAH$ and we have

$$\frac{\sin i}{\sin CAG} = \frac{\frac{EH}{AE}}{\frac{CG}{AB}} = \frac{AB}{AE} \tag{215}$$

Since $EH = CG$. Therefore CAG is the angle of refraction.

156. Refraction at a Plane Surface. Consider a luminous point O (Fig. 89), in the denser medium. A ray emerging along the line OB, will be refracted on entering the rarer medium as shown in the figure. Draw MO normal to the surface. Then the angle BOA equals i, and BIA equals r, in the rarer medium, consequently

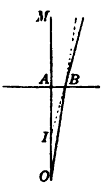


FIG. 89.

$$\frac{\sin BOA}{\sin BIA} = \frac{1}{\mu} \tag{216}$$

since the refraction is from dense to rare. Substituting from the

right triangles BAI and BAO the values of the sines we have

$$\frac{\frac{AB}{BO}}{\frac{AB}{BI}} = \frac{1}{\mu}$$

whence

$$\frac{1}{\mu} = \frac{BI}{BO}$$

or

$$\mu = \frac{BO}{BI}$$

If the point B approach very near to A, we may let $BO = AO$, and $BI = AI$, whence

$$\mu = \frac{AO}{AI} \qquad (217)$$

If the media be air and water, then a point situated at O would appear to be at I, to an observer directly over it. But μ from air to water is 4/3, hence AI is 3/4 AO, or the object appears to be but 3/4 its real distance below the surface. Conversely, to an eye under water, the point M would appear to be 4/3 its real distance away.

157. Critical Angle. When light passes from a denser to a rarer medium as from water to air, it is refracted away from the normal, the angle of refraction being greater than the angle of incidence. As the angle of incidence increases, the angle of refraction increases more rapidly, until for a certain limiting angle of incidence in the denser medium, the angle of refraction becomes 90°, or the refracted ray grazes the surface of separation between the two media. Thus in Fig. 90, the ray RO, is refracted along the line OS, while for the ray LO the refracted ray becomes OB. Obviously light incident at any angle greater than LON' can not emerge from the water, but will be reflected back again. Thus a ray meeting the surface along the line IO will be reflected to I', and since no light can emerge from the water at this incidence, it suffers *total internal reflection.*

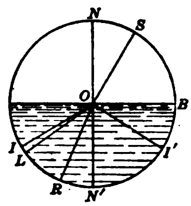


FIG. 90.

The angle LON' is called the critical angle, since for this

angle of incidence refraction ceases. The critical angle may be defined as *that angle of incidence in the denser medium, which corresponds to an angle of refraction of 90°.* If a be the critical angle, and μ the index of refraction for the two media then,

$$\frac{\sin a}{\sin 90^{\circ}} = \frac{1}{\mu} \tag{218}$$

or

$$\sin a = \frac{1}{\mu} \tag{219}$$

That is, *the sine of the critical angle is the reciprocal of the index of refraction.*

The critical angle for water is $\sin^{-1}(3/4) = 48° 27' 40''$. This means that a diver on the bottom of the ocean can see out through a cone whose base rests in the surface of the water and whose semi-aperture is 48° 27′ 40″. If he look toward the surface at an angle greater than this, he can see only the reflection of objects upon the bottom. Not only this, but since all light that could reach him must come in through this same cone, he would see all the stars in the heavens crowded into a circle whose center is the zenith, and whose angular radius is 48° 27′ 40″.

Familiar examples of total reflection are seen in the case of a tumbler of water held above the head; the under side of the upper surface of the water gives a clear, mirror-like image of the objects below it;—also in the case of a bubble of air in water, or a test tube containing air immersed in water, which when viewed at a certain angle will seem to be filled with quicksilver.

A beautiful illustration of total reflection is furnished by a stream of water flowing from a tank (Fig. 91), having a lens in the side opposite the orifice, by means of which a strong beam of light may be concentrated within the jet. The light strikes the surface of the jet at an angle greater than the critical angle and is held prisoner within the stream, until, after repeated reflection it strikes the bottom of the receptacle where it shows a bright spot. A goblet held in the stream is filled with light. The electrical illumination of fountains depends upon this principle.

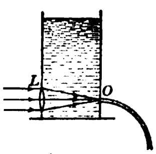


Fig. 91.

CHAPTER XVIII

PRISMS AND LENSES

158. Refraction Through a Prism. A prism is a transparent substance bounded by two plane surfaces meeting at an angle. Light on entering the prism at I (Fig. 92), is refracted *toward* the normal, along the line IE. On emerging at E the ray is refracted *from* the normal, along EO. The ray is *thus bent toward the base of the prism in each case.* Let i and r denote the angles of incidence and refraction at the first face of the prism, and r' and i' the angles of incidence and emergence at the second face. Then, since the angle between the faces of the prism is the same as that between the normals MP and NP, it follows that

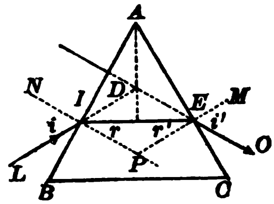


FIG. 92.

$$r + r' = A.$$

Also the deviation at the first face is $i - r$, and at the second face $i' - r'$, or the total deviation D, is

$$D = i - r + i' - r' = i + i' - A.$$

It may be shown both from theory and by experiment that for every prism there is a minimum value of the deviation, below which it cannot fall, and that this deviation is a minimum when the path of the ray through the prism is symmetrical, or when

$$i = i' \text{ and } r = r'.$$

Upon this condition we have

$$2r = A$$

or

$$r = \frac{A}{2}$$

and

$$2i = A + D$$

or

$$i = \frac{(A + D)}{2}$$

Substituting,

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin \frac{1}{2}(A + D)}{\sin \frac{1}{2} A} \qquad (220)$$

This formula is used in determining the indices of refraction of transparent solids in the form of prisms, and of liquids enclosed in a hollow prism with glass sides. The angle D denotes the angle of minimum deviation for the color under investigation.

159. Prisms of Large and Small Angle. In prisms of very small angle, we may substitute in the foregoing formula instead of the sines of the angles, the angles themselves, and the approximate formula becomes

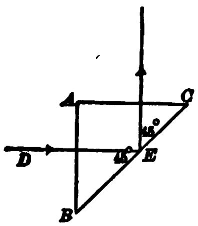


FIG. 93.

$$\mu = \frac{A + D}{A}$$

and

$$D = A\ (\mu - 1). \qquad (221)$$

If a perpendicular be dropped from A (Fig. 92), upon IE, when $r = r'$, then each half of the angle A is equal to the angle of refraction r. Now, since r must in all cases be less than the critical angle for the substance of the prism, it is clear that the angle A, cannot exceed twice the critical angle if the light is to emerge from the prism.

In the case of crown glass the critical angle is about 41° 16′. If a prism of crown glass be cut in the form shown in Fig. 93, the refracting angle A, will exceed the critical angle, and a ray DE, incident normally upon the face AB will be totally reflected at E, and emerge normally from the face AC. Such prisms are frequently employed in optical apparatus.

160. The Abbe-Littrow Principle. If the prism ABC (Fig. 92), were split along the perpendicular from A upon IE,

and the new surface were polished and silvered, it is easily seen (Fig. 94), that the refracted ray would strike the silvered surface normally, retrace its path, and return along its original direction. The ray would have traversed the half prism twice and hence would have undergone the same changes as though it had emerged from the opposite side.

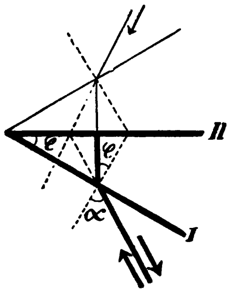


FIG. 94.

In the Abbé Spectroscope, light from a slit in the focal plane of a telescope emerging from the objective as parallel rays, is allowed to fall upon such a half prism and the returning rays are received again into the telescope, where the refracted image of the slit is made to coincide with the slit itself. By this ingenious device the instrument is much simplified and the labor of making measurements is greatly reduced, since when the refracted image of the slit coincides with the slit *the position of minimum deviation is attained.*

In this case we have

$$\mu = \frac{\sin \alpha}{\sin \phi} \tag{222}$$

161. Refraction Through a Thin Lens. A lens is a portion of a refracting medium bounded by two surfaces of revolution which have a common axis called the *axis of the lens.* By a *thin lens* is meant one whose thickness is small in comparison to its radii of curvature.

In Fig. 95, let O be the center of curvature of the spherical surface AB, forming the front surface of a thin lens. Let U be a point source of monochromatic light, of index μ. Let one ray meet the lens at A and a second ray at P. After refraction the ray UP will, if produced backward, meet the first ray in V. The points U and V are then conjugate foci *for the first surface.* From P draw the radius through O and produce it to meet

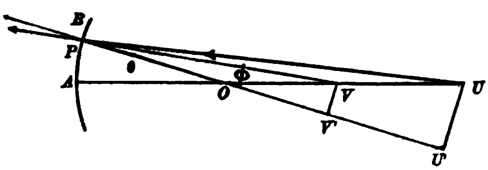


FIG. 95.

perpendiculars let fall from V and U. Then $i = UPO$, and $r = VPO$, and by the law of refraction we have

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin UPO}{\sin VPO} = \frac{UU'}{PU} + \frac{VV'}{PV} \tag{223}$$

or

$$\mu = \frac{UU'}{VV'} \cdot \frac{PV}{PU}$$

But from similar triangles

$$\frac{UU'}{VV'} = \frac{OU}{OV} \tag{224}$$

whence

$$\mu = \frac{OU}{OV} \cdot \frac{PV}{PU} = \frac{OU}{OV} \cdot \frac{AV}{AU} \tag{225}$$

if P be taken very near to the point A.

If now $AU = p$, $AV = p'$, and $AO = r_1$ we have

$$\mu = \frac{p - r_1}{p' - r_1} \cdot \frac{p'}{p} \tag{226}$$

whence

$$\frac{\mu}{p'} - \frac{1}{p} = \frac{\mu - 1}{r_1} \tag{227}$$

If for μ we substitute -1, this formula reduces to that for the concave spherical mirror, since for reflection μ must be unity and the negative sign indicates a reversal of the direction of the ray.

If now the refracted ray meet the second surface of the lens, whose radius is r_2, and whose center of curvature lies on OA, it will be refracted a second time, passing from a denser to a rarer medium. The new index of refraction will therefore be $1/\mu$. The point image V now becomes the object, and p' the object distance. Let q be the distance of the final image from the second surface and neglect the thickness of the lens.

Then

$$\frac{\frac{1}{\mu}}{q} - \frac{1}{p'} = \frac{\frac{1}{\mu} - 1}{r_2} \tag{228}$$

or

$$\frac{1}{q} - \frac{\mu}{p'} = \frac{1 - \mu}{r_2} \tag{229}$$

Adding (227) and (229) we have

$$\frac{1}{q} - \frac{1}{p} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{230}$$

This formula is to be understood as an approximation, giving accurate values only in the case of pencils of small angular opening.

162. Discussion of Formula. If the source be at an infinite distance, formula (230) of the last article becomes

$$\frac{1}{q} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{231}$$

If we set this value of q equal to f, we may write

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{232}$$

where f is called the *focal length* of the lens, *i. e.,* the distance from the rear surface of the lens to the focus of rays parallel to the *principal axis.* Such a focus is called the *principal focus,* its conjugate point lies at infinity.

163. The Sign of the Quantity f. It is proposed to determine when the focal length of a lens is to be considered positive and when negative. Consider the meniscus-shaped lens BA (Fig. 96). Let $dc = T$, the thickness of the lens, measured parallel to the axis at a distance $cn = y$, from the axis. Let $t = BA$, the thickness of the lens at the axis; *i. e.,* the value of T for $y = 0$.

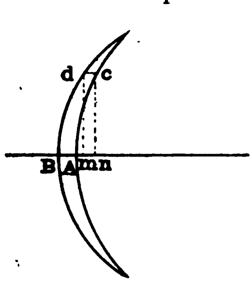


FIG. 96.

Let r_1 and r_2 be the radii of the first and second surfaces respectively. Then, neglecting An^2,

$$An = \frac{y^2}{2r_1}, \tag{233}$$

and

$$Bm = \frac{y^2}{2r_2}$$

Also

$$Bm + T = An + t \tag{234}$$

or

$$T - t = An - Bm = \frac{y^2}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \tag{235}$$

whence

$$T - t = \frac{y^2}{2(\mu - 1)} \cdot \frac{1}{f} \tag{236}$$

Hence, since μ is greater than unity, the sign of f is positive or negative according as T is $>$ or $< t$; that is, according as the lens is *thicker at the edges than at the center* or *vice versa.*

A lens whose focal length is *positive,* that is, a lens thicker at the edges than at the center, and whose principal focus lies on the same side of the lens as the source of light, is called a *concave* or *diverging* lens. A lens thicker at the *center* than at the edges, has its focal length *negative,* and is called a *convex* or *converging* lens. Examples of the various forms of convex and concave lenses are shown in Fig. 97.

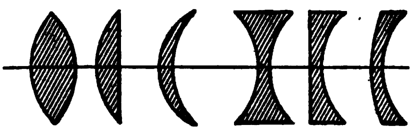


FIG. 97.

In case the two radii r_1 and r_2 are equal, the expression

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

reduces to

$$-\frac{1}{f} = (\mu - 1)\frac{2}{r_1} \tag{237}$$

for a double convex lens. From this it is evident that r_1 is negative, as it should be, since *the center of the first surface lies on the opposite side of the lens from the source of light.*

In the case of a double concave lens we have

$$\frac{1}{f} = (\mu - 1)\frac{2}{r_1} \tag{238}$$

or r_1 is *positive* in this case, and *the center of the first surface lies on the same side as the source.*

164. Discussion of Lens Formula. Since in the expression

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

the right hand member contains only constant quantities for any

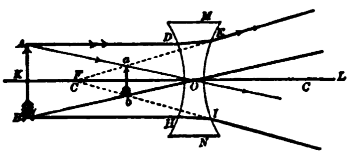
given lens, it follows that the focal length of a lens is a constant, and depends simply upon the material of the lens and upon the curvature of its surfaces. Again, since

$$(\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \text{ is equal to both } \frac{1}{f} \text{ and to } \frac{1}{q} - \frac{1}{p}$$

we may write

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{f} \qquad (239)$$

It has already been shown that for a concave lens f is always positive. Now since p, the object distance, is essentially positive, it follows that q *must be positive and less than* p. This means that that image lies on the same side of the lens as the object, that it is virtual and nearer the lens than the object. This is shown in Fig. 98, where the concave lens MN, produces a virtual, diminished, erect image *ab*, of the object AB, and the distances KO, kO and FO, represent the quantities p, q and f.

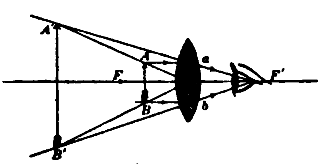

FIG. 98.

In convex lenses it has been shown that f is negative and the formula becomes

$$\frac{1}{q} - \frac{1}{p} = -\frac{1}{f} \qquad (240)$$

Three cases require consideration:

(a) *When* $p < f$, or $1/p > 1/f$, q must be positive and greater than p, and image and object lie on the same side of the lens. The image is then virtual, erect, and magnified as shown in Fig. 99. In this case if we call the intersections of image and object with the axis of the lens, k and K, we have $KO = p$, $kO = q$, and $FO = f$. An example of this case is seen in a single lens used as a simple magnifier.


FIG. 99.

(b) *When* $p > f$, or $1/p < 1/f$, q must be negative, since $1/q - 1/p$ is equal to a negative quantity. This means

that the image lies on the opposite side of the lens from the object, that it is real, inverted, and smaller or larger than the object according as p is greater or less than p. This case is illustrated in Fig. 100.

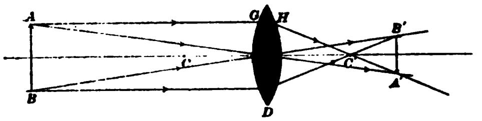


FIG. 100.

Since, for a real image in the case of a convex lens, q is negative, we may rewrite the formula after changing signs,

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{f} \tag{241}$$

(c) *When* $p = q$, we have

$$\frac{2}{p} = \frac{1}{f}, \text{ or } p = 2f \tag{242}$$

This means that the image is at the same distance behind the lens as the object is in front of it, that it is inverted, real and the same size as the object. This case is applied in a method for measuring the focal length of a convex lens. The lens, an object, and a screen are mounted upon an optical bench and the screen moved until the image is the same size as the object. The distance from object to screen is then four times the focal length of the lens.

It remains to be noted that in all cases of images formed by lenses, *the size of the image is to the size of the object, directly as their distances from the center of the lens.*

165. Image and Object at a Fixed Distance. It appeared in the previous article that the real image formed by a convex lens may be larger or smaller than the object. If a convex lens, an object and a screen be mounted upon an optical bench and the distance between object and screen be made more than four times the focal length of the lens, the two images may be thrown upon the screen in succession by simply moving the lens. In the first case the lens is nearer the object and the image is correspondingly magnified. In the second case the two distances p and q are interchanged and the lens is nearer the image. If we call the distance

from the object to screen l, and the difference in the two settings of the lens a, then

$$\begin{aligned} q+p&=l \\ q-p&=a \end{aligned} \qquad (243)$$

whence

$$q=\frac{l+a}{2} \text{ and } p=\frac{l-a}{2}$$

substituting in

$$\frac{1}{q}+\frac{1}{p}=\frac{1}{f}$$

we have

$$\frac{1}{f}=\frac{1}{\frac{l+a}{2}}+\frac{1}{\frac{l-a}{2}}$$

or

$$f=\frac{l^2-a^2}{4l} \qquad (244)$$

This is a more accurate method of measuring f than that given in the previous article, since it is difficult to focus accurately in the case where the two images coincide, *i. e.*, where $l=4f$.

166. Constants of Thick Lenses. In the case of lenses whose thickness cannot be neglected, recourse is had to the method of Gauss, who first demonstrated that the formula

$$\frac{1}{q}-\frac{1}{p}=\frac{1}{f}$$

may be applied to thick lenses through the introduction of certain cardinal points to be determined for each lens. The constants of a thick lens are defined as follows:*

Conjugate planes are planes normal to the axis of a system, so related to each other that to every point in the plane in object space there corresponds a point in the plane in image space.

Magnification. If y and y' be the respective distances of a point object and its point image from the axis of the system, then the ratio y'/y is called the ***magnification, m.***

* For a detailed discussion of this subject the student is referred to Müller-Pouillet's Lehrbuch der Physik, 9th ed. vol. 2, part I, pp. 85-180, or Drude's Theory of Optics, pp. 17-30.

The principal planes eH, e'H' (Fig. 101), are the two conjugate planes in every system, in which object and image are of the *same size* and *stand in the same position.* Since in the principal planes the magnification

$$m = \frac{y'}{y} = +1 \tag{245}$$

the principal planes are sometimes called *unit planes* or planes of *unit magnification.* The points of intersection H and H' of the principal planes with the axis of the system are called the *principal points,* or *unit points.*

The *focal points F,* and *F'* are the points on the axis in which all rays parallel to the axis in image and object space respectively, meet after passing through the system.

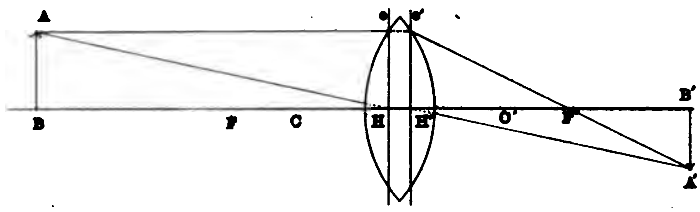


Fig. 101.

The *focal planes* are the two planes normal to the axis at the focal points, and having their conjugates at infinity.

The *focal lengths f* and *f'*, of a system for object and image space respectively, may be defined as the distances from the focal points F and F', to the principal points H and H'.

If in Fig. 101 we set $BH = p$, $B'H' = q$, $H'F' = f$, then our previous formula

$$\frac{1}{q} - \frac{1}{p} = \frac{1}{f}$$

for convex lenses still holds. This means that the distances from object to lens and image to lens are now to be measured to the principal points instead of to the surface of the lens as heretofore.

In the case of crown glass lenses of equal curvature the principal points are located within the lens, at one-third the thickness of the lens from the curved surfaces. In the case of plano-concave or plano-convex lenses of crown glass, one of the

principal points lies on the curved surface and the other at one-third the thickness of the lens from the curved surface. In thin lenses the two principal points coincide at the center of the lens. In lenses of the meniscus form one or both principal points may lie outside the lens.

167. Geometrical Significance of Focal Lengths. In any lens system (Fig. 102), let F and F', H and H', represent the focal points and principal points, in object space and image space respectively. Let AP, a ray through F, the focal point in object space, make an angle u with the axis, and let $A'P'$ be the corresponding or conjugate ray in image space. In general, parallel rays in object space must intersect in some point in the focal

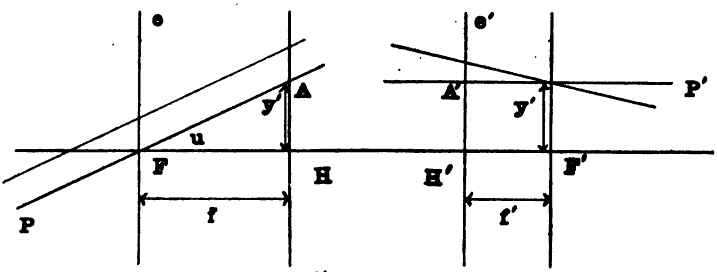


FIG. 102.

plane through F' in image space. Let this point be distant y' from the axis. The value of y' evidently depends upon the angle of inclination u, which the incident ray makes with the axis. If $u = o$, then $y' = o$, that is, rays parallel to the axis intersect in F'.

But in the case of ray PFA which passes through F, the focal point in object space, and cuts the principal plane H in A, u is not zero. Its conjugate ray $A'P'$, must evidently be parallel to the axis, and, from the definition of principal planes, A and A' are equidistant from the axis.

Hence y', the distance from the axis to the image formed by a parallel beam incident at an angle u, is shown from the figure to be

$$y' = f \tan u, \tag{246}$$

and by symmetry we may write

$$y = f' \tan u', \tag{247}$$

where u' is the angle under which a ray parallel to the axis in object space, cuts the axis in image space.

168. Gauss' Definition of Focal Lengths. Let P (Fig. 103), represent the position of an infinitely distant object, *e. g.*, the sun, from the system S. The rays proceeding from all points of the sun may be regarded as parallel. The system S receives therefore, simply cylinders of rays which are focused in the corresponding points of the rear focal plane through F'. This produces in F' a small image of the sun. Let h' represent its diameter, and let u be the angle which the extreme rays from the sun form with the axis. Then the *front focal length*,

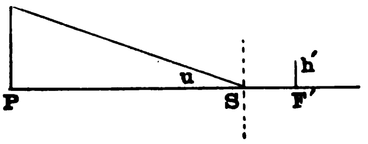


FIG. 103.

$$f = \frac{h'}{\tan u} \qquad (248)$$

is equal to the diameter of the sun's image divided by the tangent of the angle subtended by the sun's disk. This latter angle is termed the *visual angle* or the *apparent magnitude*.

The focal length may therefore be defined as follows:

The focal length of object space is the quotient obtained by dividing the linear magnitude of the image of an infinitely distant object by the tangent of the angle subtended by that object, or

$$f = \frac{y'}{\tan u} \qquad (249)$$

For the rear focal length we may write

$$f' = \frac{y}{\tan u'} \qquad (250)$$

or *the focal length of image space is equal to the distance between the axis and any ray parallel to it in object space, divided by the tangent of the inclination of its conjugate ray.*

169. Determination of Focal Lengths. Let LL' *and* QQ' (Fig. 104), be two pairs of conjugate planes with reference to a system whose focal points are F and F'.

Let the two rays I and II intersect at P and their conjugates I' and II' at P'. Ray I is parallel to the axis in object space and passes through F' at an angle u'. Ray II passes through F at an angle u, and is parallel to the axis in image space. The

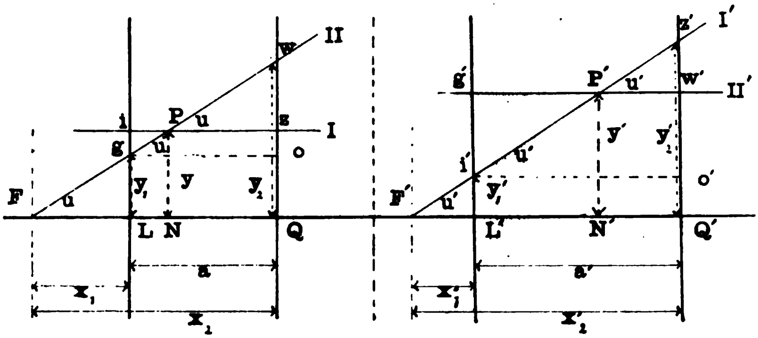


FIG. 104.

planes L and L' may be considered as the first pair and Q and Q' as the second pair of conjugates, to which the subscripts of x and y correspond.

Through a special choice of rays we have for ray I in object space, $iL=zQ=PN$, or $y_1=y_2=y$, and consequently the magnification may be expressed by

$$\frac{y'_1}{y}=m_1, \text{ and } \frac{y'_2}{y}=m_2$$

In a similar manner we have for ray II in image space $g'L'=w'Q'=P'N'$, or $y'_1=y'_2=y'$, and

$$\frac{y_1}{y'}=\frac{1}{m_1},\ \frac{y_2}{y'}=\frac{1}{m_2} \tag{251}$$

From the triangle wgo, we have

$$\frac{ow}{og}=\tan u=\frac{y_2-y_1}{a} \tag{252}$$

multiplying both numerator and denominator by y', we have

$$\tan u=\frac{y'}{a}\cdot\frac{y_2-y_1}{y'} \tag{253}$$

but, since

$$\frac{y_2}{y'} = \frac{1}{m_2}, \text{ and } \frac{y_1}{y'} = \frac{1}{m_1}$$

therefore

$$\tan u = \frac{y'}{a}\left(\frac{1}{m_2} - \frac{1}{m_1}\right)$$

or

$$\frac{a}{\frac{1}{m_2} - \frac{1}{m_1}} = \frac{y'}{\tan u} = f = \frac{x_2 - x_1}{\frac{1}{m_2} - \frac{1}{m_1}} \quad (254)$$

In a similar manner from triangle $z' i' o'$,

$$\tan u' = \frac{y'_2 - y'_1}{a'} \quad (255)$$

or

$$\tan u' = \frac{y}{a'} \cdot \left(\frac{y'_2 - y'_1}{y}\right) = \frac{y}{a'}(m_2 - m_1) \quad (256)$$

whence

$$\frac{a'}{m_2 - m_1} = \frac{y}{\tan u'} = f' = \frac{x'_2 - x'_1}{m_2 - m_1} \quad (257)$$

In practice a lens system is set up and the image of an object of known dimensions, is measured for two positions of the object L and Q, whose distance apart, $a = x_2 - x_1$, can be accurately determined. From these data the values of m_1 and m_2 can be computed and inserted in formula (254).

In the case of a microscope objective, the magnification of a known object is determined, first with the draw tube of the microscope pushed in, and next with it drawn out through a known distance a', the positions of the images corresponding to L' and Q' (Fig. 104). The resulting values of m_1 and m_2 are then substituted in formula (257). This principle finds application in the Abbé Focometer and in related methods of determining the focal lengths of lens systems.

X **170. Spherical Abberation.** It is to be noted that the formula for spherical mirrors as well as that for refraction through a thin lens is accurate only for pencils of small angular aperture. In cases where the opening of the mirror or lens is no longer small, the focus for parallel rays is no longer a point. Rays from points at a distance from the axis of the reflecting or refracting

surface, are brought to a focus sooner than those rays lying nearer the axis. This results in confusion and distortion of the image, which is called *spherical aberration.*

In the case of astronomical specula where a large reflecting surface is essential for light gathering purposes, the spherical form can not be used. The surface of the mirror must be of parabolic section, since in the case of the parabola any ray parallel to the axis passes after reflection exactly through the focus. In compound lenses the defect of spherical aberration is avoided by combining two or more spherical surfaces so that their respective aberrations may mutually annul each other. In a simple lens the defect is partially removed by the use of a diaphragm which stops off the rays from the edges of the lens, thus leaving only the central part of the lens effective.

Since the image of a star is always a point on the axis of the lens, the axial aberration is of chief importance in astronomical work. In photographic lenses, however, it is necessary to recognize and correct for ***five different kinds of spherical aberration,*** and the manufacture of high grade photographic lenses becomes correspondingly complicated. The discussion of the formulæ by means of which these corrections are effected is too difficult for an elementary text.

PROBLEMS

1. Two plane mirrors are placed parallel and facing each other at a distance of 20 cm. Required the distance of the first three images, in each mirror, of an object placed 8 cm. from one of the mirrors.

Ans. 8, 32, and 48 cm. from first; 12, 28, and 52 cm. from second.

2. A mirror is made to revolve about a vertical axis 25 times a second. If a horizontal beam of light is allowed to fall on the mirror from a fixed source, required the velocity at which the reflected beam would traverse a circle 78 cm. in diameter having its center on the axis of the mirror.

Ans. $1.23(10)^4$cm./sec.

3. Let *AB* and *CB* be two mirrors inclined to each other at an angle, x. Also, let p' be the image in *AB* of any point, p,

placed between the mirrors, and p'' the image of p' in CB. Show that the angular separation of pBp'' is twice the angle between the mirrors.

4. An object 0.96 cm. long is placed at a point 35 cm. in front of a concave mirror having a focal length of 30 cm. Required the size and position of the image.

Ans. 5.76 cm. long; 210 cm. in front.

5. What is the radius of a spherical mirror which forms an image at a distance of 46.2 cm. in front of the mirror when the object is placed 153 cm. from the vertex.

Ans. $R = -71.0$ cm.

6. What will be the size of the image of the sun formed by a mirror having a radius of 275 cm., the diameter of the sun being taken as 32 min.? *Ans.* 1.28 cm.

7. If an eye immersed in a fluid whose index of refraction is 1.42, look out through a horizontal surface, what will be the greatest apparent zenith distance of a star? *Ans.* $44° 46'$.

8. Find the radius of the circle on the upper surface beyond which light-waves, emitted by a luminous point at the bottom of a layer of liquid 4.2 cm. deep and having an index of refraction of 1.25, will cease to emerge. *Ans.* 5.594 cm.

9. When a layer of liquid 4.65 cm. deep is poured upon a dot on a glass plate, the position of its image, as found by the necessary change in the focus of a microscope, is 1.37 cm. above the plate. What is the index of refraction of the liquid?

Ans. $\mu = 1.41$.

10. What would be the minimum deviation produced by a prism whose angle is 1.3°, for which $\mu = 1.54$?

Ans. 45.4′

11. The minimum deviation produced in monochromatic light by a prism whose angle is 45.05° was 26.67°. What is the index of refraction? *Ans.* $\mu = 1.530$.

12. The radii of curvature of a thin double convex lens are 46.4 cm., and the index of refraction 1.53. What is its focal length? *Ans.* $f = 43.8$ cm.

13. Required the focal length of a thin lens which forms an image at a distance of 30.3 cm. behind the lens, when the object is placed 91.1 cm. in front. *Ans.* $f = 22.7$ cm.

CHAPTER XIX

DISPERSION

171. Dispersion and Recomposition of Light. When a ray of sunlight is admitted through a narrow vertical slit into a darkened room and passed through a long focus lens, a sharply defined, white image of the slit is projected upon a screen placed near and parallel to the opposite wall of the room. If now a glass prism with its refracting edge parallel to the slit be placed just inside the focus of the lens, the light is refracted from its original direction toward the base of the prism, and is spread out into a wedge-shaped beam with its apex at the prism and its base upon the screen, which should now be moved so as to stand normally to the refracted light. The original white image of the slit has now been replaced by a series of overlapping, colored images, forming a continuous band of brilliant color, in which the tints vary insensibly from red nearest the first position of the slit image, through all shades of orange, yellow, green, blue and indigo, to violet at the end of the band nearest the prism.

Such a band of color is called *a spectrum.* If sun light be used it is called a *solar spectrum.* The separation of white light into its colors is called *dispersion.*

The dispersion of light was first explained by Newton in 1666, who announced as the result of his experiments, that "rays of differently colored light have different degrees of refrangibility." He therefore concluded that white light is to be regarded as a mixture of an endless series of tints, among which the so-called "colors of the rainbow," violet, indigo, blue, green, yellow, orange and red were selected by him as distinctive. He also proved that after light has been dispersed by passing it through a prism, it suffers no further change on passing through a second prism.

Newton further showed that if the various colors of the spectrum be recombined by any means, as by reflecting them

from a series of mirrors, or by receiving them upon a concave mirror and focusing them upon a single point, the result was white light as in the original source.

This experiment may be readily performed by allowing the light from the first prism to traverse a second, similar prism (Fig. 105), with its refracting edge turned in the opposite direction. If the prisms be exactly similar and the adjustment be correct, the image of the slit falls in its original position and is perfectly white. The effect of the combined prisms is thus seen to be equivalent to that of a plate with parallel sides.

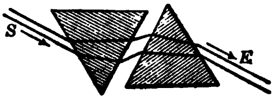


FIG. 105.

In accordance with the physical explanation of refraction to be given later, it may be shown that the varying refrangibility of the different colors is due to the different retardations experienced by these colors in traversing the denser medium. It has been conclusively proven that in a denser medium the speed of violet light is less than that of red light. In its passage through the prism therefore, the violet light is most retarded and is consequently most refracted, while the red light is least deviated from its original direction. For any two media the index of refraction is, therefore, not a constant but increases from the red to the violet end of the spectrum.

172. The Frauenhofer Lines. If the slit in the foregoing experiment be made narrower the overlapping of the spectral images is diminished, since the narrower the slit the smaller is the required difference in refrangibility between two adjacent colors in order to separate their corresponding images in the spectrum. Wollaston in 1802, was the first to use a narrow slit as a source of light and to secure an approximately *pure spectrum,* that is, a spectrum in which the various colors are almost completely separated from each other. He used a slit about one-twentieth of an inch wide, which he viewed at a distance of 10 or 12 feet, through a prism held near the eye. By thus receiving the spectrum directly into the eye he was able to detect certain dark lines crossing the solar spectrum, which he believed to mark the limits of the spectral colors.

In 1814, Faunhofer, working independently of Wollaston, rediscovered these dark lines, made a map showing their relative positions, and designated the more prominent lines by letters of the alphabet. Thus, (Fig. 106), the *A* line (not shown in figure), lies in the extreme red, the *B* and *C* lines in the medium and lighter red, the *D* line in the yellow, the *E* in the green, *F* in the greenish-blue, *G* in the deep-blue, while the two *H* lines mark the limit of the violet end of the spectrum for the eyes of most observers.

Fraunhofer also used a single prism at a distance from a narrow slit, but he received the spectrum into a telescope, which was first sharply focused upon the slit before the prism was



FIG. 106.

put in place. By this means he was enabled to observe with such accuracy that he counted 754 lines between *B* and *H*, and located with certainty the positions of 350 of them upon his map.

The Fraunhofer lines in the solar spectrum are of extreme importance in the study of optics, both from a practical and a theoretical point of view. They afford a ready and accurate means of designating lights of definite colors, and are constantly used as reference lines in the determination of refractive indices. On the other hand they indicate the partial absence of certain colors in the light of the sun, and thus not only offer a starting point for the study of the nature of light, but lead to important conclusions concerning the constitution and condition of the sun itself.

173. Total, Mean, Partial, and Relative Dispersion. If sunlight be passed through three prisms having the same refracting angle, one of flint glass, one of crown glass, and one a hollow prism with plane glass sides and filled with water, the resulting spectra will be found to differ greatly in length. Thus (Fig. 107), the spectrum from the flint glass prism is about twice as long as that from the one of crown glass, and three times as long as

that from the water prism. It is clear that the various colors undergo widely different deviations by prisms of the same angle but of different substance. A careful study of the dispersion of various refracting media is therefore a prerequisite for the scientific construction of optical instruments.

If $\mu_B, \mu_C, \ldots \mu_H$, represent the refractive indices of a given prism for the corresponding Fraunhofer lines, then $\mu_H - \mu_B$, represents the *total dispersion* for that prism in the spectral region from B to H. Likewise the differences $\mu_C - \mu_B$, $\mu_D - \mu_C$, $\mu_F - \mu_D$, etc., are termed the *partial dispersions* for the corresponding regions. Since the middle region $C - F$, includes the brightest part of the spectrum, the difference $\mu_F - \mu_C$, is termed the *mean dispersion*. Also, since the D line lies near the brightest part of the spectrum, its index μ_D, is called the *mean index* for a given substance. The ratio of the mean dispersion to the quantity $\mu_D - 1$, is termed the *relative dispersion* or *dispersive power*, $\left(\frac{\mu_F - \mu_C}{\mu_D - 1}\right)$, and is used to characterize different grades of optical glass with reference to the possibility of selecting achromatic combinations.

From Article 159 we have seen that for prisms of small

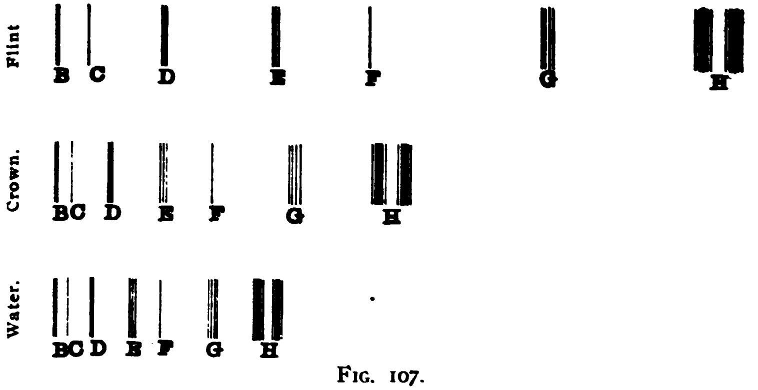


FIG. 107.

refracting angle A, the deviation D, for any color is defined by the equation

$$D = A\,(\mu - 1).$$

Obviously the angular dispersion ψ, for any two lines as B and H, is given by the difference between their respective deviations, or,

$$\psi = A\,(\mu_H - \mu_B) \tag{258}$$

It remains to be noted that since different glasses differ so widely in *relative dispersion,* it is within the power of the optician to produce at will, prism combinations that will give *either deviation without dispersion or dispersion without deviation,* according as the need for each may arise. The discussion of these combinations will follow later.

174. Irrationality of Dispersion. If, through variation of the refracting angles, the spectra from the flint glass, crown glass and water prisms should be made of exactly the same length, it would be found that the colors are not equally dispersed by the three substances. If the three spectra be arranged one above the other, and so adjusted that the B and H lines coincide, it will be evident (Fig. 108), that the spectra are quite different.

Thus in the water spectrum, the interval from B to F is equal to that from F to H. In the crown glass spectrum the

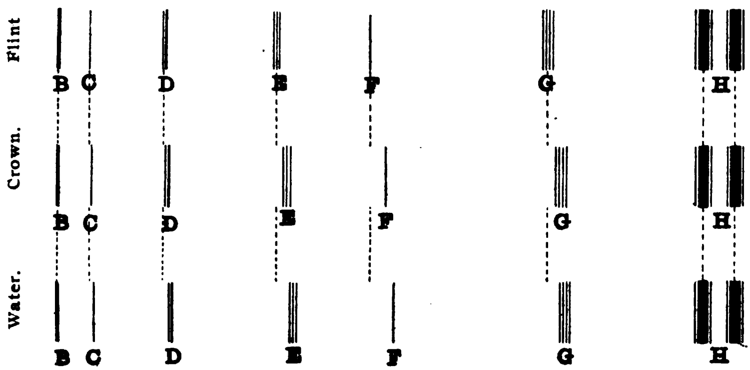


FIG. 108.

interval from B to F is smaller than that from F to H, and in the flint glass spectrum it is smaller still. It thus appears that in flint glass the dispersion is relatively greater in the blue end of the spectrum, while for water the dispersion is relatively greater in the red end.

If prisms of other substances should be examined still other variations in the dispersion of the various colors would be found. In short there is no relation between the change in the refractive index with the wave-length in different substances. This peculiarity is termed the *irrationality of dispersion.* It is due to this fact that the spectra produced by different prisms cannot readily be compared with each other. Herein lies a fundamental difference between *prismatic spectra and spectra produced by gratings.* In grating spectra, as we shall see later, the deviation for every color and for every grating is directly proportional to the wave-length. Hence any grating spectrum may at once be compared with any other. On this account the spectra from gratings are termed *normal spectra.*

175. Anomalous Dispersion. A more surprising peculiarity of dispersion was discovered by Christiansen in 1870, who found for a hollow prism filled with an 18.8 per cent solution of fuchsine, or aniline red, an entirely new succession of colors in the spectrum. In this case the violet light was refracted the least, then came the red and then the yellow; the green and bluish green were wholly absorbed. Such a spectrum is termed *abnormal,* and the substance is said to exhibit *anomalous dispersion.*

Through the investigations of Kundt, it has been demonstrated that anomalous dispersion is characteristic of all substances possessing "surface color," *i. e.,* substances which exhibit a different color by reflected light from that shown in transmitted light. Thus fuchsine is reddish violet by transmitted light but an intense green by reflected light. All bodies of this class exhibit total reflection for certain colors of the spectrum at all incidences, and their solutions even when very dilute, show marked absorption bands in the same colors.

Kundt observed anomalous dispersion in the solutions of a large number of substances among which were the red, blue, green and violet anilines, carmine, indigo, orsellin, litmus, hematine, chlorophyll, iodine in carbon disulphide, and the extracts of red wood and sandal wood. Among the solids cobalt blue glass shows anomalous dispersion very well.

It was theoretically proven by Helmholtz, and subsequently confirmed by the experimental work of Pflüger, Wood, and Magnusson, that in all cases where a substance presents one or more

absorption bands in its spectrum, the refractive index of the substance for colors immediately below the absorption band is increased enormously, while the index for colors just above the band is correspondingly depressed. The index for the color that is wholly absorbed becomes infinite. From this point of view the apparent "anomaly" in the dispersion of strongly absorbing substances disappears. Fuchsine absorbs the green, hence the refractive indices of the red and yellow are increased and that of the violet diminished; the violet is therefore least refracted and the yellow the most. It is now well established that anomalous dispersion is exhibited by all gases and vapors showing absorption bands in the spectrum.

176. Chromatic Aberration. Since it has been shown that the refractive index of a substance differs for different colors, being greatest for violet and least for red, it is clear from the formula

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

that the focal length of a lens is shortest for the violet and longest for the red light. In other words, if a simple lens (Fig. 109) be presented to a parallel beam of white light, the violet rays will come to a focus first at *v*, and the red last at *r*, the foci for the remaining colors being distributed along the axis between these points. In consequence of its dispersion therefore, a single lens instead of producing for each point object a *single point image*, in reality produces a *series of colored images*. Hence the image of any object formed by a single lens in white light is surrounded by fringes of color and is generally blurred and indistinct. This defect of lenses is called *chromatic aberration*. An image in which all the colors are focused at the same point is *colorless* or *achromatic*.

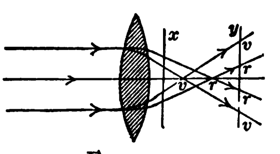


FIG. 109.

About 1670 Newton concluded from his experiments upon the refractive indices of glass and water, that the relative dispersion of all substances *was a constant*, and he consequently despaired of being able to make a lens that would give a colorless image. It was not until 1758 that the London optician, John Dollond, succeeded in making such a lens.

We have seen that light which had been dispersed by one prism could be reunited into an achromatic image by a second, similar prism with its refracting edge turned in the opposite direction. In this case the second prism corrected *both the deviation and the dispersion.* If now, the second prism be made from a glass having a *high relative dispersion,* as flint glass, then the dispersion of the crown glass prism may be corrected by a prism of *much smaller angle,* and an *outstanding deviation* will thus be left. The condition for achromatism in the two prisms is obtained by equating their angular dispersions, or

$$A\ (\mu_H - \mu_B) = A'\ (\mu'_H - \mu'_B)$$

whence

$$\frac{A}{A'} = \frac{\mu'_H - \mu'_B}{\mu_H - \mu_B} \qquad (259)$$

or, *for achromatism the angles of the two prisms must be inversely proportional to the dispersions of the two prisms in the region to be achromatized.*

In an exactly similar way it is possible so to combine a convex lens of crown glass and a concave lens of flint glass that their resulting dispersion shall be zero and the system shall still have any desired focal length. In *lenses* the *curvatures* of the four surfaces are varied, as the *angles* have been in the case of the two *prisms,* but since there are *four* radii to be chosen, two may be adjusted for achromatism and the remaining two so selected as to reduce spherical aberration to a minimum.

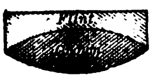


FIG. 110.

Such a combination (Fig. 110) is called an *achromatic lens,* and is said to be corrected for some two colors. If the lens is to be used for visual work it is usually corrected for the region C to F. If the combination is to be applied to photography the achromatization is computed for the region G to H, since in this region the chemical action of light is most pronounced.

Owing to the irrationality of dispersion it is obviously impossible to achromatize any region perfectly unless the "course" of the dispersion of the two glasses is the same over that region. The slight residual color defects in an achromatic pair are termed "secondary spectra," and may be reduced still further by the addition to the combination of a third lens of the proper curva-

ture and dispersion. By the use of the new optical glasses introduced by Abbé and Schott of Jena, it is now possible to achromatize more perfectly with two lenses than it had formerly been possible to do with three.

177. Direct Vision Spectroscope. If two prisms, one of flint and the other of crown glass, have their angles so chosen that their *deviations* for some one color shall be equal, that is, so that

$$A\ (\mu - 1) = A'\ (\mu' - 1), \tag{260}$$

there will remain an *outstanding dispersion* due to the flint prism. This fact has been utilized in the construction of the direct vision spectroscope. Usually the instrument contains two, ninety degree, flint glass prisms combined with three prisms of crown glass.

Fig. 111 shows a combination of one dense flint prism P, and two crown glass prisms. By means of the achromatic lens O, the slit S, is sharply focused for sodium light. On turning the instrument toward some source of white light the image of

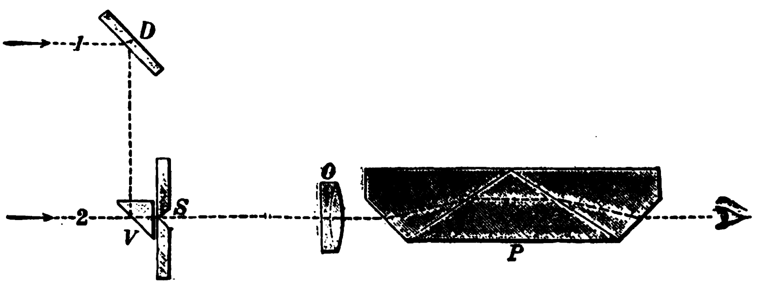


FIG. 111.

the slit is spread out into a spectrum. The width of the slit S, is adjustable, and by making it narrow the Fraunhofer lines are readily seen in ordinary daylight. Light from the source under examination is admitted directly into the instrument by the slit, (ray 2), while by means of the comparison prism V, and the mirror D, light from a second source (ray 1), may be introduced, and the spectra of the two sources observed side by side. The deviation of the prism train is usually adjusted to zero for the D line.

PROBLEMS

1. If the indices of refraction for the four substances, flint glass, crown glass, water, and carbon disulphide are

	B	*C*	*D*	*F*	*H*
Flint glass	1.6127	1.6144	1.6193	1.6315	1.6527
Crown glass	1.5301	1.5311	1.5339	1.5404	1.5509
Water (18.7°C.)	1.3310	1.3320	1.3336	1.3380	1.3448
CS_2 (18.7°C.)	1.6182	1.6219	1.6308	1.6555	1.7020

compute the angle of a flint glass prism necessary to achromatize the region from *B* to *H* in a crown glass prism whose angle is 3°30′.

2. From above table compute the angular separation of the lines *C* and *F* produced by a hollow prism filled with carbon disulphide, of angle 40°, if the slit image be half an inch wide, and the prism be 30 feet from the screen.

3. In a direct vision spectroscope (Fig. 111), the flint glass prism has an angle of 54°. Compute the base angles of the two crown glass prisms needed to give direct vision for the *D* line, using data given above.

4. Compare the lengths between *B* and *H*, of spectra produced by two hollow prisms of angle 40°, one filled with water and the other with carbon disulphide. Prism distant 20 feet from screen.

5. *AB* is the diameter of a polished semicircular are *APB*. A ray of light proceeds from a point *Q* in the tangent at *A*, and after reflection at *P* and *B* returns to *Q*. If the length of the ray's path be 2 feet, the mirror's diameter is very nearly 7.35 inches.

6. A person whose hight is *h*, observes vertically beneath his eye, an object at the bottom of a clear pool: he then removes to a distance *d*, keeping his eye on the object, when his line of vision makes 45° with the surface; show that if $\mu^2 = 2.5$, the depth of the pool $= 2\ (d - h)$.

CHAPTER XX.

OPTICAL INSTRUMENTS

178. Projection Apparatus. The projection lantern consists essentially of two optical systems, one for condensing light upon the object, and the other for forming the image upon the screen. The source of light is usually either the lime light or the arc lamp. This source is placed just outside the focus of the "condenser," which generally consists of two plano-convex lenss of short focus with their convex surfaces turned toward each other. The cone of light emerging from the condenser falls upon the object, a transparent drawing or lantern slide, and then passes through the projecting lens or objective. This is usually an achromatic lens of about one foot focal length. Since the object is just outside the focus of the objective, the image projected upon the screen is real, inverted, and highly magnified.

179. The Camera Obscura. In the *camera obscura* we have the projection apparatus reversed. It consists, as its name indicates, of a darkened chamber having an opening on one side, in front of which is placed a projecting lens. Upon the wall of the chamber opposite the lens is a white screen for receiving the image of illuminated objects which chance to be in front of the lens. The image is real, inverted and smaller than the object. This instrument was formerly used as an aid in sketching buildings or the prominent features of a landscape. Since the discovery of the various methods of fixing an image upon a photographic plate by the chemical effects of light, it has developed into the photographer's camera.

Th simple lens has been replaced by a more or less elaborate optical system carefully corrected for spherical and chromatic aberrations. The sides of the chamber have been made flexible to admit of focussing the image accurately, and the size of the aperture is carefully adjusted to the existing degree of illumin-

ation by a series of graded diaphragms. In its present form, the photographer's camera is one of the most delicate, accurate and important of optical instruments, revealing many things that escape the eye, since it secures the cumulative effect of faint luminous impressions.

180. The Eye. From an optical point of view, the eye is a photographic camera with an automatic adjustment for focus, and a sensitive plate that reports the images directly to the brain. Fig. 112 represents a sectional view of the human eye. The *cornea a*, and the *sclerotic* coat *b*, form the outer coating of the darkened chamber, a nearly spherical cavity about one inch in diameter. The sclerotic coat is tough and strong, holding the ball of the eye in shape, and forms the "white of the eye."

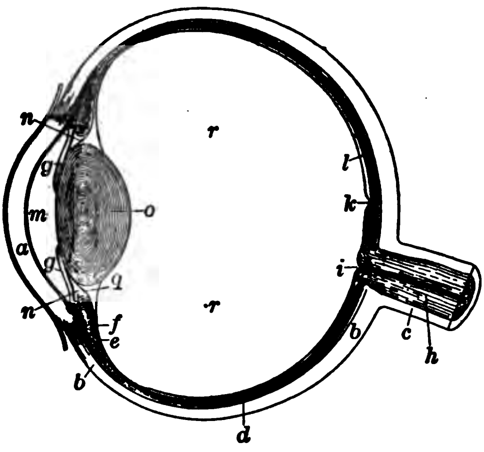


FIG. 112.

The cornea is a transparent, horn-like membrane, which serves to admit light into the eye and to retain the *aqueous humor m*, in a lenticular form.

The second coat is the *choroid d*, a black, opaque membrane which serves to darken the interior of the eye. In front, the choroid coat gives place to the *iris gg*, an opaque, colored diaphragm, which regulates the amount of light entering the eye through the circular opening in its center, which is called the pupil of the eye. Just back of the iris lies the *crystalline lens o*, a double convex lens of unequal curvatures, composed of a transparent, jelly-like substance built up in concentric layers. The optical density of these layers increases as we go toward the center. The interior cavity *r*, is filled with a viscous mass of fluid of the highest transparency, called the *vitreous humor*.

The third coat is the *retina l,* a delicate membrane formed by the expansion of the optic nerve, which serves as the sensitive screen for the reception of the images formed by the optical system of the eye. The posterior surface of the retina is characterized by a mosaic-like structure of nerve endings, termed "rods and cones." These microscopic little bodies stand perpendicular to the retinal surface, are closely crowded together, and each one is furnished with a separate nerve fiber. These are believed to form the real *light sensitive layer* of the retina.

The retina possesses two spots of peculiar interest. The "*yellow spot,*" situated in the optical axis of the eye, marks a small area where the retinal layer is slightly thickened. In its center is located the *fovea centralis k,* where the sensitive layer is reduced to an exceeding thinness, only the most minute nerve filaments extending into it. This is the spot of *finest space discrimination.* It is upon this spot that the image of anything is brought which we wish to examine attentively. The point *i,* at which the optic nerve enters the eye is entirely insensible to light, and is called the *"blind spot."*

The adjustment of the crystalline lens for focussing objects at varying distances is accomplished by changing the thickness of the lens itself, through the action of the *ciliary muscle e.* This power of adaption or "accommodation" as it is called, is most marked in young children, and gradually diminishes through life. At about the fortieth year the eye has lost so much of its power of accommodation as to require the aid of glasses to focus small objects near at hand.

A normal eye can accommodate for distinct vision for objects at all distances from 6 inches from the eye (near point), up to infinity (far point). Within this range of accommodation, however, there is a point whose distance from the eye is termed the "distance of distinct vision." It is defined as the distance at which a normal eye can most readily read ordinary print, and is assumed to be *25 centimeters or 10 inches.*

Since the formation of images by the eye, is identical with that of the photographic camera, it follows that the images perceived by the retina are inverted. The ability of the eye and brain to interpret inverted images as belonging to erect objects

is probably gained by experience, through combination with other sense perceptions, especially with that of touch.

181. Defects of Vision. Through structural defects the eye may be unable to accommodate for distant objects. The lens in this case is too powerful and the focus falls *in front* of the retina instead of upon it. Such an eye is called *"short-sighted"* or *myopic*. The myopic eye requires a concave lens to counteract the excessive refraction of the eye and bring the focus upon the retina, as shown in Fig. 113.

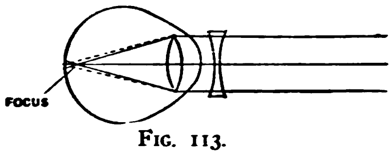


Fig. 113.

Again, the power of the lens may be below the normal, in which case the focus for parallel rays lies *behind* the retina instead of upon it. Such an eye is termed *hypermetropic* or *"far-sighted."* For such an eye a convex lens is needed to enable the eye to bring the focus of parallel rays upon the retina, as shown in Fig. 114.

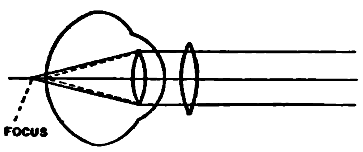


Fig. 114.

The eye gradually loses its power of accommodation during life and at about the fortieth year it becomes unable to accommodate for objects near at hand, although the vision for distant objects is as good as ever. Such an eye is said to be *presbyopic*. For such eyes convex glasses are to be used for reading and for examining objects close at hand.

It frequently happens that the eye has different power in different planes. A bright point is seen not as point but as a line. Horizontal, vertical or diagonal lines, like the spokes of a wheel or the letters on a clock dial, may be seen with unequal distinctness. Such a defect is called *astigmatism*. For such eyes glasses having cylindrical surfaces are used.

182. Apparent Size and Magnification. The angle subtended by an object at the optical center of the eye is called the *visual angle*. The apparent size of a linear object is measured by the visual angle which it subtends. Thus if d is the distance

of an object from the eye, b the distance of the retinal image from the optical center of the crystalline lens, δ and β the lengths of object and image respectively, then V, the visual angle, or the apparent size of the object is

$$V = \frac{\delta}{d} = \frac{\beta}{b} \tag{261}$$

Hence *the apparent size of a linear object is inversely proportional to its distance from the eye.* Also the size of the retinal image of an object is inversely proportional to the distance of the object from the eye, and directly proportional to the size of the object.

The clearness with which the minute details of an object can be distinguished increases with the size of the retinal image. Hence to see a thing clearly we bring the object up to the near point of the eye and thus secure the maximum efficiency of the unaided eye. The effect of any optical instrument which may be used to assist the eye, consists simply in increasing the apparent magnitude of the object. Hence the magnifying power of any instrument may be defined as the ratio of the apparent magnitude *through* the instrument, to the apparent magnitude *without* the instrument, or

$$\frac{\text{Apparent magnitude through instrument}}{\text{Apparent magnitude without instrument}} = \text{Magnification.}$$

183. The Simple Microscope. The simple microscope is a converging lens of short focus. When an object is placed slightly nearer to the lens than its focal distance, an eye brought close up to the lens perceives a virtual, erect and magnified image $A'B'$, as shown in Fig. 115. If we conceive the eye to be accommodated for infinite distance, then the object is moved up to the focus F, and the image moves off to infinity. The object is now seen clearly by the eye although much nearer to the eye than would have been possible for distinct vision except for the lens. The microscope thus enables the eye to see an object clearly at a much shorter distance, and hence acts as an aid to the accommodation for this distance.

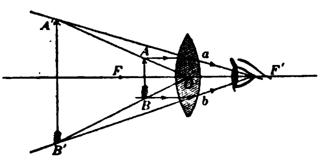


FIG. 115.

The visual angle under which the object appears *through the lens*, is $\frac{AB}{f}$, where AB is the length of the object and f is the focal length of the lens. The visual angle subtended by the object *without the lens*, is $\frac{AB}{25 \text{ cm.}}$; hence the magnification m. produced by the lens for an eye adjusted for infinity is

$$m = \frac{AB}{f} \div \frac{AB}{25 \text{ cm.}} = \frac{25 \text{ cm.}}{f} \qquad (262)$$

or *the magnification due to a simple lens is obtained by dividing the distance of distinct vision by the focal length of the lens.*

To a certain degree the effect of magnification may be obtained without the use of a lens, by looking at a brilliantly illuminated object through a very small hole, as through a pin-hole in a card. The hole acts as a diaphragm or second pupil to the eye, and thus, by stopping off the rays from the edges of the pupil, permits of the formation of an image at a distance much less than the normal 25 cm. Such magnification is obtained however, at the cost of illumination, while the lens accomplishes the same result without reducing the illumination.

184. The Astronomical Telescope. In the astronomical telescope (Fig. 116), the objective *L*, is a converging lens of long focal length *F*, which forms a real inverted image A_1B_1 of a very

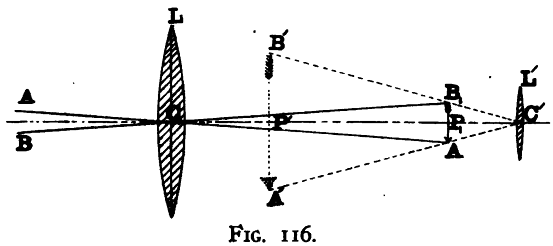


FIG. 116.

distant object. This image subtends the same angle at *C*, the center of the objective, as does the object itself. Hence $\frac{A_1B_1}{F}$ represents the apparent magnitude of the object.

The real image formed by the objective is viewed through the eye-piece *L'*, used as a simple magnifier. The eye-piece

forms a virtual, magnified image $A'B'$, of the image A_1B_1. The apparent magnitude of the object as seen *through the telescope* is therefore, the visual angle $B'C'A'$, or $\frac{A_1B_1}{f}$, where f is the focal length of the eye-piece. The magnification m, of the telescope is therefore given by the equation

$$m = \frac{A_1B_1}{f} \div \frac{A_1B_1}{F} = \frac{F}{f} \qquad (262)$$

or *the magnification of the telescope is obtained by dividing the focal length of the objective by the focal length of the eye-piece.*

185. The Compound Microscope. The compound microscope in its simplest form may be regarded as a telescope used for examining objects near at hand. The object glass O, (Fig. 117), is a converging lens of very short focus placed near the object AB, but still outside its principal focus. The real, inverted image $A'B'$ is much magnified. This image is viewed by the eye-piece E, also a converging lens, used as a simple magnifier, which forms the virtual, highly enlarged image $a\,b$.

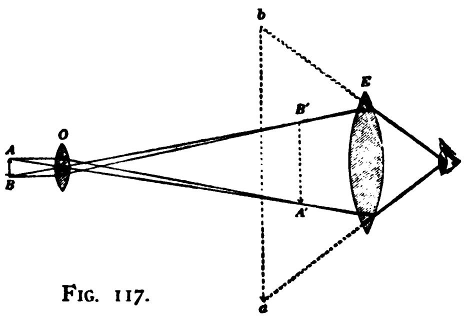


FIG. 117.

The approximate magnification may be written down at once. Letting o represent the center of the objective, we have for the magnification of the object glass,

$$\frac{A'B'}{AB} = \frac{B'o}{Bo} = \frac{L}{F}$$

where L is the length of the microscope tube and F the focal length of the objective. Also, since the magnification by the eye-piece is $\frac{25 \text{ cm.}}{f}$, the total magnification is

$$\frac{L}{F} \times \frac{25}{f} = \frac{25\,L}{Ff} \qquad (263)$$

The practical advantage of the compound microscope consists in the possibility of using higher powers. With the simple magnifier the working distance soon becomes too small to enable one to observe with comfort or accuracy. Thus with a magnification of 100, the lens must be brought within 0.1 of an inch of the object and the eye must approach the lens very closely indeed if none of the rays are to fall outside the pupil. At the same time the dimensions of the lens become too minute for accuracy in grinding.

186. Spectroscope and Spectrometer. It has been noted that the conditions for obtaining a pure spectrum were approximated by Fraunhofer in his arrangement for observing the dark lines in the solar spectrum. The one disadvantage consisted in the loss of light due to the distance between the slit and the prism. In the modern spectroscope (Fig. 118), this loss is avoided by the device of the *collimator*. The collimator is simply a telescope in which the eyepiece is replaced by an adjustable slit. When in adjustment, the slit is brought into the focal plane of the achromatic objective, so that all light entering the slit leaves the objective as parallel rays. In this way the full intensity of the light from the slit is preserved and the additional advantage of parallelism of the rays is secured.

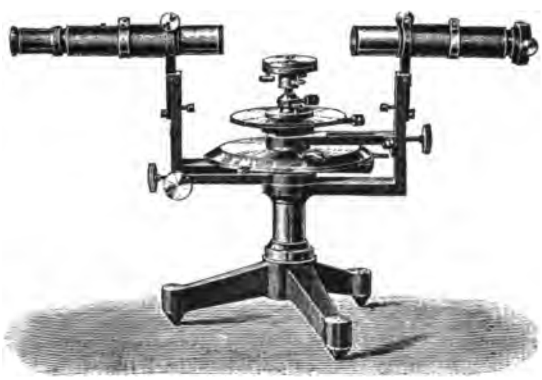


Fig. 118.

The remaining parts of the spectroscope are a suitable table for mounting the prism and an observing telescope with achromatic objective, for receiving the spectrum. The telescope moves about an axis concentric with the prism table. When the instru-

ment is furnished with a graduated circle for determining accurately the positions of telescope and prism, it becomes a *spectrometer,* and may be applied to a variety of optical measurements.

In the Abbé spectrometer (Fig. 119), the functions of the



FIG. 119.

collimator and the observing telescope are combined. A small, totally reflecting prism furnished with an adjustable slit is inserted in the focal plane of the eye-piece of the observing telescope, at one side of the field of view. Light admitted through a small window in the side of the telescope tube, is reflected by the prism through the slit, directly into the optical axis of the telescope and leaves the objective as parallel rays. By the use

of the Abbé prism described in Article 160, the dispersed light is returned upon its path and the spectrum is viewed by the telescope in the unobstructed part of the field. For the position of minimum deviation the refracted image of the slit is made to coincide with the slit itself and the settings can be made with great exactness. By means of the micrometer screw *M*, the dispersion of any substance in the various spectral regions is measured independently of the graduated circle, thus greatly reducing the labor of making the measurements and at the same time securing a marked increase in the accuracy of the results. An auxiliary telescope permits of the use of the instrument in the ordinary form.

CHAPTER XXI.

VELOCITY OF LIGHT

187. Velocity of Light—Roemer's Method. The first definite proof of the finite velocity of light was given by Roemer, a Danish astronomer in 1676, from a study of the eclipses of the satellites of the planet Jupiter. This planet has several satellites which revolve about it as our moon does about the earth. The inner one of these satellites has a mean period of 42 h. 28 m. 36 sec. As it passes behind the planet it disappears from view quite suddenly and the interval between two eclipses can be determined with considerable accuracy. Roemer noticed that this interval was not constant throughout the year; that it was *less* than the period of revolution of the satellite when the earth was *approaching* Jupiter, and *greater* than this period when the earth was *receding* from Jupiter.

Thus (Fig. 120), the period of the satellite coincided with the eclipse interval when the earth and Jupiter occupied the positions $E'J'$, or $E\ J$, but as the earth passed from E' to E, the eclipses occurred systematically *ahead* of the computed time by differences whose sum at E amounted to 1000 seconds. From E to E' they *fell behind* by the same amount. Roemer concluded that this difference in time must be required by the light to traverse the difference in path $E\ E'$ between these two positions. Assuming the diameter of the earth's orbit to be 186,000,000 miles, the velocity of light was computed to be 186,000 miles per second.

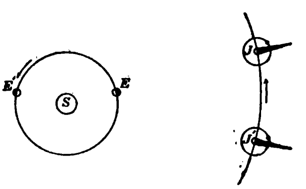


FIG. 120.

Roemer's wonderful discovery was received with little favor by the scientific world and was practically disregarded for nearly

50 years, until Bradley's discovery of the aberration of light, in 1728, gave an additional method for measuring this important physical constant.

188. Velocity of Light—Foucault's Method. In 1850, Foucault, a French physicist, described a method for the direct determination of the velocity of light, which in its improved form has given the most accurate values yet attained. Foucault's original arrangement consisted of an illuminated slit *S*, (Fig. 121), from which light passed through a lens *K*, of long focus to

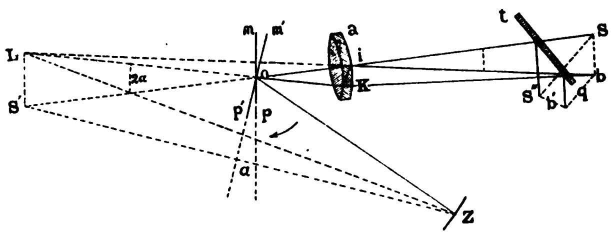


FIG. 121.

a mirror *mp*, which was capable of rapid rotation about an axis through *o*, normal to the plane of the paper. From this rotating mirror the light was reflected to a concave mirror *Z*, striking it normally, since the center of curvature of *Z* lay in *o*. The mirror *Z* and the slit *S* lay in the conjugate foci of the lens *K*. The light was therefore returned upon its path and after a second reflection at the rotating mirror, fell again upon the slit. By interposing a piece of plane parallel glass *t*, at an angle of 45° to the path of the light, a displaced image *S''*, of the slit was produced for the purpose of observation. When the rotating mirror *mp*, was put in motion, a flash of light fell upon the concave mirror at each revolution and for rotations of more than 30 per second these flashes blended into a steady image of the slit.

Now, if in the time *t*, needed for the light to travel the distance 2*D*, from the rotating mirror to the concave mirror and back again, the rotating mirror should have turned through a small angle, then the reflected image of the slit would have been displaced through twice that angle. If, therefore, we call the distance from the rotating mirror to the slit *r*, and the displacement of the slit *s*, then from a knowledge of the quantities *D*,

s, r, and n, the number of rotations of the mirror per second, the velocity of light V, can be calculated. Thus

$$t = \frac{2D}{V} \tag{264}$$

Also, if the mirror turn through a small angle α in time t, then

$$\alpha = \omega t = 2\pi n t \text{ or } t = \frac{\alpha}{2\pi n} \tag{265}$$

Now the light is turned through an angle $2\alpha = s/r$, where the displacement of the slit image is small, and

$$\alpha = \frac{s}{2r} \tag{266}$$

From (265) and (266) we have

$$t = \frac{s}{4\pi n r} \tag{267}$$

and combining (264) and (267) we get

$$\frac{s}{4\pi n r} = \frac{2D}{V} \tag{268}$$

or finally,

$$V = \frac{8\pi n r D}{s} \tag{269}$$

In Foucault's experiment D was 20.24 meters, r was 1.0257 meters, the speed of the mirror was 400 revolutions per second and the measured displacement s, of the slit image was 0.7 mm. From these values he computed the velocity of light to be

$$V = 298{,}000 \frac{km.}{sec.}$$

In Foucault's experiment the displacement, 0.7 mm, was a quantity too small to be measured with accuracy, while the speed of rotation of the mirror was enormous. In 1878, Michelson so modified the method of Foucault as to render it capable of far greater accuracy. The lens L (Fig. 121a), was so placed that the rotating mirror m, lay very near its principal focus. All rays from the slit s, therefore, left the lens as approximately parallel rays. For

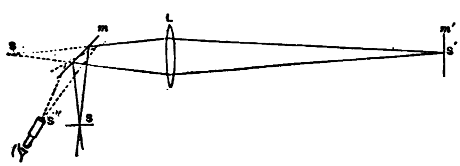


FIG. 121a.

the concave mirror, a plane mirror m', was substituted and the distance D was increased to 600 meters, instead of 20 meters as in Foucault's experiment. In this way Michelson succeeded in securing a displacement of the slit of 13 cm., or 133 millimeters, without using a speed of rotation for the mirror of more than 200 per second. The mean of Michelson's measurements gave the velocity of light.

$$V = 2.999 \times 10^{10} \frac{cm.}{sec.}$$

Newcomb also using the method of the rotating mirror obtained a value in close agreement with this.

The experiment of Foucault was originally designed for the purpose of actually measuring the velocity of light in media of different density in order to decide experimentally between the corpuscular and the undulatory theories of light. According to the corpuscular theory, light should travel faster in a dense medium than in a rare one; from the principles of the undulatory theory exactly the opposite conclusion was reached. Experiment alone could decide. Foucault in 1850 measured the velocity of light in air and in water and found the velocity in water to be less than in air. Since that time the emission theory of light has been definitely abandoned and the undulatory theory is now generally accepted in some form or other among scientific men. A statement of the theory forms the topic of the next article.

189. The Undulatory Theory of Light. According to the undulatory theory light consists in an extremely rapid vibratory motion which is transmitted from point to point in the form of transverse waves. Since experience shows that light traverses a space the more readily the less ponderable matter the space contains, it is concluded that light may be propogated even in space containing no ponderable matter, *i. e.,* in a vacuum. It is assumed that the universal medium for the transmission of luminous disturbances is the ether, which consequently can not be ponderable matter, although it must possess many properties in common with it.

In order to formulate the undulatory theory it is only necessary to assume that the ether fills all space, that it has different density in different media, and that the laws of elasticity are

absolutely the same in it as in ponderable matter. Moreover, as the ether interpenetrates the molecules of ponderable matter, its particles receive and transmit vibrations in the same way and according to the same laws as do the molecules of ponderable matter. In following out the consequences of these assumptions it is necessary to bear in mind certain results previously established for wave motions in general.

(*a*). When, in an elastic medium of density d and coefficient of elasticity e, a molecule is displaced by force, the general equilibrium of the medium is destroyed; all the neighboring molecules experience a movement which is propogated from point to point in all directions with a velocity

$$V=\sqrt{\frac{e}{d}} \tag{270}$$

This same law is held to apply to the ether where V is the velocity of light.

(*b*). The intensity of any vibratory disturbance proceeding from a point source is *directly proportional* to the *square of the amplitude,* and *inversely proportional* to the *square of the distance from the source.*

190. Equations of Wave Motion. If, in place of an infinitely small and instantaneous movement, a particle of ether execute regular vibrations its oscillations, if simple harmonic, may be expressed by the equation

$$y = A \sin \frac{2\pi t}{T} \tag{271}$$

in which y is the displacement of the ether particle at any time t, A is the amplitude and T the period.

If light be transmitted with a velocity V, from an ether particle P_1 to a second particle P_2 distant x from P_1, the time required for transit is x/V. Now if equation (271) represent the condition at P_1 then the condition at P_2 is represented by

$$y' = A' \sin\left(2\pi \frac{t - x/V}{T}\right) \tag{272}$$

since P_2 is always in a given condition of vibration, x/V seconds *after* P_1 has been in the same condition. The difference in the two vibrations is, as may be readily seen, a difference in *ampli-*

tude and a difference in *phase.* If the wave be generated at a point source in a homogeneous and isotropic medium then the disturbance travels outward in all directions with the same velocity. Hence all points on the surface of a sphere with center at P, and radius x, must be in the *same phase.* Surfaces containing only points in the *same phase of vibration* are called *wave surfaces.* The distance from one wave surface to the next is called a *wave-length* λ, and is defined in terms of velocity and period of vibration by

$$\lambda = VT.$$

If we introduce λ into equation (272) we have for the condition at P_2,

$$y' = A' \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad (273)$$

191. Superpositions of Small Vibrations. Whenever an ether particle is actuated at the same time by impulses due to two sets of vibrations, the resultant motion is that due to the *superposition* of the *two vibrations.* Since in all cases the amplitudes of vibrations are assumed to be small the treatment of the problem is termed the superposition of *small* vibrations. If we represent the two impulses actuating the ether particle by the two equations

$$y_1 = A_1 \sin \frac{2\pi t}{T}$$
$$y_2 = A_2 \sin\left(\frac{2\pi t}{T} - \frac{2\pi x}{\lambda}\right) \qquad (274)$$

and for convenience set

$$\frac{2\pi t}{T} = a, \text{ and } \frac{2\pi x}{\lambda} = \beta,$$

then the resultant motion of the ether molecule may be represented by

$$Y = y_1 + y_2 = A_1 \sin a + A_2 \sin (a - \beta). \qquad (275)$$

Expanding the right hand member and adding we have

$$Y = (A_1 + A_2 \cos \beta) \sin a - A_2 \sin \beta \cos a;$$

or putting

$$A_1 + A_2 \cos \beta = A \cos \phi$$

and

$$A_2 \sin \beta = A \sin \phi \qquad (276)$$

we get

$$Y = A \sin (a - \phi) \qquad (277)$$

This shows that the resultant motion of the ether particle is still a simple harmonic motion, with an amplitude A. The resultant amplitude may be found at once from equations (276) by squaring and adding, whence

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos \beta$$

or, replacing β by its value $\frac{2\pi x}{\lambda}$ we have

$$A^2 = A_1^2 + A_2^2 + 2\,A_1\,A_2 \cos \frac{2\pi x}{\lambda} \tag{278}$$

This equation shows that the square of the resultant amplitude, and hence the ***resultant intensity,*** due to the superposition of two vibrations, depends upon the quantity x, that is, upon the ***difference in path*** traversed by the two component vibrations. If $x = 0$, or *an even multiple of* $\lambda/2$, then the resultant amplitude is the *sum* of the amplitudes; if $x = \lambda/2$, $3\lambda/2$, $5\lambda/2$, etc., then the resultant amplitude is the *difference* of the component amplitudes. These results are applied in the theory of interference.

192. Law of Reflection of Light Deduced from Huygens's Principle. Let AB (Fig. 122), be a plane wave front meeting the reflecting surface CD, in the direction indicated. In accordance with Huygens's principle, all points on the reflecting surface become new centers of disturbance, as the successive points in the wave front reach them. Consequently the point A, being the first disturbed, acts as a center from which the disturbance spreads backward into the original medium with the original

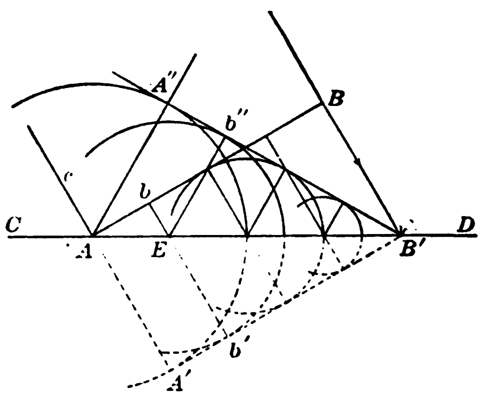


FIG. 122.

velocity, and in the time t needed for the disturbance at B to reach B' it has spread through a hemisphere with radius $A A'$ equal to $B B'$ of which the full line above CD is a trace on the plane of the paper.

In the same time the disturbance at b on the incident wave front has reached E, from which point the new wave spreads back in the hemisphere whose radius is $E b'$. In a similar way every point on the reflecting surface has generated new waves. The radius for the wave from B' is of course zero at the end of the time t. Now a tangent from B' upon the circle $A'' A'$ will touch all the other subsidiary circles; consequently the reflecting surface $C D$ has given rise to the new wave front $B' A''$, which now moves off parallel to itself in the original medium.

By the undulatory theory the angle of incidence is the angle included between the *incident wave front* and the *surface, i. e.,* the angle BAD; similarly the angle of reflection is the angle $AB'A''$ included between the *reflected wave front* and the *reflecting* surface. The equality of these angles can be shown at once from the equality of the triangles ABB', $AB'A'$, and $AB'A''$.

193. Law of Refraction of Light Deduced from Huygens's Principle. In the case of refraction of light it must be remembered that the ether imbedded between the material particles of the different media will seem to have different densities and hence will transmit the luminous disturbance with different velocities. Thus suppose the plane wave front AB (Fig. 123), in medium I, to meet the plane surface AC of medium II, in the direction BC. Now if the second medium should transmit the luminous impulses with the same velocity V, as the first, then CD' would be the refracted wave front, that is there would be no refraction at the surface of separation.

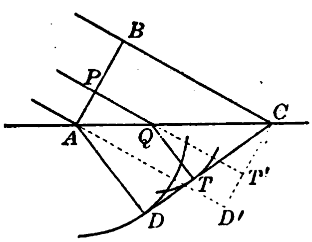


FIG. 123.

But if the second medium transmit light more slowly than the first, say with a velocity V', then in time t while the light in the first medium is passing from B to C, the light in the second

medium will have spread out into a hemisphere about A as a center, and having a radius AD, such that

$$\frac{AD}{AD'} = \frac{V'}{V}, \text{ or } AD = \frac{V'}{V} \cdot AD'$$

The circular arc through D about A, having its center at A, represents the trace of this hemisphere upon the plane of the paper. In the same way from the point Q, there spreads out a hemispherical disturbance which at the end of the time t, has reached a radius

$$QT = \frac{V'}{V} \cdot QT'$$

A tangent from C upon the wave front about A as a center, is also tangent to the entire system of subsidiary waves, and DC is therefore the refracted wave front, which now moves on in the new medium parallel to itself. By a previous definition the angle of incidence is the angle BAC or its equal ACD'. Similarly the angle of refraction ACD, is the angle included between the *refracted wave front* and the *refracting surface.*

Applying the law of refraction

$$\frac{\sin i}{\sin r} = \frac{\sin ACD'}{\sin ACD} = \frac{\frac{AD'}{AC}}{\frac{AD}{AC}} = \frac{AD'}{AD} = \frac{V}{V'} = \mu \qquad (279)$$

It thus appears that *when light passes from a medium A to a medium B* the relative *index of refraction for the two media is equal to the ratio of the velocities of light in A and B.*

PROBLEMS

1. If the nearest distance for distinct vision for a far-sighted person is 2 ft. 11 in., what should be the focal length of the spectacles he would require for reading? *Ans.* $f = 14$ in.

2. If the greatest distance of distinct vision for a myopic eye is 3.9 in., what should be the focal length of the proper reading spectacles? *Ans.* $f = 6.4$ in.

3. A converging lens placed at a distance of 5.2 cm. from

a luminous object forms an image on a screen. When the lens is moved a distance of 23 cm. nearer the screen, another image is formed. What is the focal length of the lens?

Ans. $f = 4.4$ cm.

4. A ray of light strikes a plane parallel plate of glass of index of refraction 1.5, at an angle of 70°, and emerges from the other side parallel to its original direction, but displaced laterally through a distance of 5 millimeters. How thick is the glass?

5. A man 6 feet tall stands upon a smooth, level walk 300 feet distant from an observer. A boy 4 feet tall standing between them seems to be of the same hight as the man. How far away is the boy if the eye of the observer be in the plane of the walk?

6. A telescope whose objective has a focal length of 12 feet, is furnished with two eye pieces whose focal lengths are one inch, and 0.5 in. respectively. Compute the magnifying power in each case.

7. A person of normal vision, uses a convex lens whose focal length is 2.5 cm. as a simple magnifier. What is the magnification produced?

8. The objective of a microscope has a focal length of 6 mm., the eyepiece 24 mm.; the object is 6.1 mm. from the center of the objective. Find (*a*) the position of the real image; (*b*) the magnification produced by the objective; (*c*) the magnification produced by the eye piece; (*d*) the total length of the microscope; (*e*) the total magnification.

9. A lens of 0.5 in. focal length is used to project a microscopic object upon a screen thirty feet distant from the object. Determine the magnification and the position of the lens.

10. A double convex lens of radii 80 cm. is made from crown glass whose refractive indices are given on page 200. How far apart upon the axis of the lens are the foci for the *B* and the *F* lines?

11. A projecting lantern is to be used in a lecture room, where the screen is 60 feet distant. If the required magnification be 20 find the focal length of the new objective needed.

CHAPTER XXII.

INTERFERENCE

194. General Statement. Thomas Young showed that under certain circumstances two nearly parallel beams of light do not, when superposed, produce increased illumination, but that they may even so disturb each other's effects as mutually to extinguish each other and produce darkness. In such cases the light waves are said to *interfere* and the resulting phenomena are classed under the head of *interference phenomena.*

Interference phenomena are of two general kinds: first those in which the two interfering pencils have undergone only reflections and refractions, and have had certain phase differences produced thereby; the second in which interference takes place between subsidiary waves starting from *different parts of the same wave front.* The latter phenomena are usually classed as *diffraction phenomena.*

It was shown in the study of sound that two sets of sound waves might be made to interfere and produce either a sound of increased intensity or total silence, according as the two vibratory motions were in the *same* or *opposite* phase. Similar phenomena may be produced in the case of two trains of light waves *provided the two sets of waves proceed originally from the same source.* If the two pencils proceed from two different sources, as from two flames or from different parts of *one and the same flame,* they are incapable of producing interference. Such sources are termed *incoherent.* Two pencils proceeding from the same point source are termed *coherent pencils.*

If two sources are to produce interference their phases must always be either exactly the same, or they must have a fixed difference. In the case of incoherent pencils, the sources may change their difference in phase many thousands or even millions of times in a second, and yet each interval include millions of

vibrations. Such changes prevent the appearance of interference.

It should be carefully noted, however, that in light as in sound, the system of interference bands denotes only a redistribution of the vibratory energy, and while it may be reduced to zero at some points and heaped up at others, yet the amount of energy is the same.

195. Interference from Two Small Apertures. Suppose a beam of monochromatic light from a narrow slit fall upon the two small apertures A and B (Fig. 124), so that the two pencils from these points are coherent and in the same phase. Draw OM normal to AB, at its middle point, and at M erect a perpendicular PM. Now, since M is equidistant from the two apertures, it is evident that the paths traversed by the two pencils are identical and the two wave trains will reach the point M *in the same phase,* and the illmination will be a maximum. If we assume the amplitudes to be equal then the resultant amplitude of vibration will be the sum of the two equal amplitudes and will therefore be doubled, while the intensity of illumination will be four times as great as from either source singly.

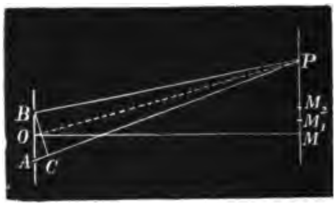


FIG. 124.

Next suppose a point M_1, be chosen such that the difference in path of the two pencils is $\lambda/2$ for light of the particular color used. Then $AM_1 - BM_1 = \lambda/2$, and the pencil from A will reach M_1 just one-half period behind the pencil from B, the vibrations will be in *opposite phase* and will destroy each other, and there will be darkness at this point. If at M_2 the difference in path amount to $2\,\lambda/2$, or a whole wave length, then the vibrations from A reach M_2 one whole period behind those from B, and hence coincide in phase, and the two sets of vibrations reenforce each other, producing maximum illumination.

In general, if P be any point, either above or below M, such that $AP - BP = \pm\, n\,\lambda/2$, then P will be a bright or dark point according as n is *even* or *odd.* If, therefore, the slits be illuminated by sodium light there will appear upon the screen normal

to the plane of the paper, a series of alternately bright and dark fringes starting with M, and extending both above and below this point. If either aperture be closed the fringe system vanishes.

The distance of any band from the central band, may be calculated as follows. Call the distance MP, x; let MO equal a, and denote the distance AB between the two sources by c. Draw the lines OP, BP and AP, and about P as center, with a radius BP describe the short arc BC. AC will therefore, represent the difference in path, $n\,\lambda/2$. The two angles ABC and POM are equal since their sides are mutually perpendicular, and the two triangles ABC and OMP are similar. Also, since the angles ABC and POM are very small their sines and tangents may be equated and we have

$$\frac{PM}{OM} = \frac{AC}{AB}$$

whence

$$x = \frac{a}{c} \cdot n \frac{\lambda}{2} \tag{280}$$

This formula shows that the distance of any fringe from the central fringe M, varies directly as the wave length of the light employed. If, therefore, white light be used, the central fringe at M, being the position of zero difference in phase, will be white. The other parts of the system instead of being marked by bright and dark bands, will now show a set of rainbow colored fringes, and there will be no dark bands at all. This is because the different colors correspond to different wave lengths. Moreover, experiment shows that the fringes are *violet* on the *inner edges nearest* M, and red on the outer edges, hence we see that violet light has the *shortest* and red the *longest* wave length of the spectral colors. If with monochromatic light, the quantities a, c, n and x be carefully determined, the wave length λ for the corresponding color can be computed.

196. Fresnel's Mirrors. In order to demonstrate the phenomenon of interference of light as clearly as possible Fresnel devised the following experiment . Two plane mirrors OM and ON (Fig. 125), of polished metal or black glass are so placed

that their two planes are vertical, and meet in a vertical line, forming an angle which differs but little from 180°. Monochromatic light from a vertical slit S, falls upon the two mirrors and is reflected in the general direction OC, as if proceeding from the two virtual images A and B. In the region upon the screen about C illuminated by the two coherent pencils, we may expect a series of interference fringes, normal to the plane of the paper.

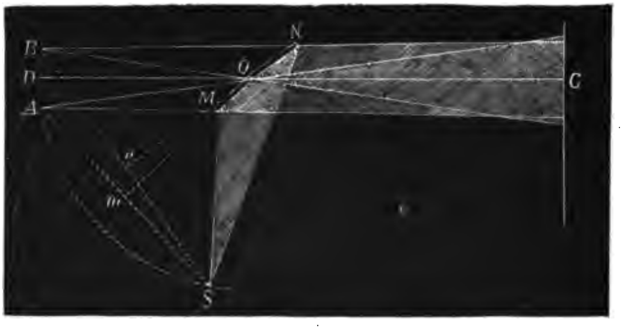


FIG. 125.

To show that the phenomenon is really due to interference, it is only necessary to cover the face of one mirror with any opaque substance when the whole system of fringes vanishes.

Fig. 125 represents a horizontal section through the mirrors and the vertical slit S, which is supposed to stand normal to the plane of the paper. Upon On and Om, the directions of the planes of the two mirrors, drop the perpendiculars Sn and Sm, and produce these perpendiculars as far behind the planes of the mirrors as S lies in front of them. Then B and A represent the virtual images of S, seen in ON and OM respectively. From the construction it is evident that B, A, and S, lie upon the circumference of a circle having O as its center. Through O draw CD normal to AB, and complete the figure.

The angles nOm and ASB are equal, since their sides are mutually perpendicular. If $nOm = \omega$, then AOB is 2ω. Let $DO = a$, and $OC = b$; then $AB = 2a \sin \omega$, and the formula of the preceding article for the distance to the nth fringe,

$$x = \frac{CD}{AB} \cdot n \frac{\lambda}{2}$$

becomes

$$x = \frac{a+b}{2a \sin \omega} \cdot n\frac{\lambda}{2} = \frac{a+b}{2a\omega} \cdot n\frac{\lambda}{2} \qquad (281)$$

since ω is a very small angle.

The distances a and b may be read off from the positions of the supports upon an optical bench, the distance x may be measured by a micrometer screw and n can be counted. In this way the wave length of any colored light may be measured.

197. Fresnel's Biprism. As a modification of the experiment with the two mirrors, Fresnel devised the following experiment for producing interference fringes by refraction. An isosceles glass prism $C E D$, (Fig. 126), with the angle at E very nearly 180°, receives monochromatic light from a vertical slit at O. The two halves of the prism CE and DE, behave as two right angled prisms put base to base, and the light from the upper half is refracted downward as if coming from a virtual source B, while that passing through the lower half of the prism gives rise to a second virtual image at A. The distance between these two



Fig. 126.

virtual images A and B, is smaller the smaller the angles C and D become, so that by making E very nearly 180°, the two pencils emerging from the two halves of the double prism (biprism) overlap throughout a certain region FMG upon the screen. M, being equidistant from the two virtual sources, is a bright band for all colors. On either side of M there appear alternately dark and bright bands, standing normally to the plane of the paper, marking the positions for which the difference of path is an odd or even multiple of $\lambda/2$ for the special wave length of light under consideration. If white light be used the dark and bright bands

are replaced by a series of rainbow colored fringes, violet on the inner and red on the outer edges.

The distance of the nth band from the center is readily calculated. Thus let OE equal a, EM equal b, ϵ equal the angle C or D of the prism, and μ its refractive index. The deviation δ produced by either half of the biprism may be written

$$\delta = \varepsilon\ (\mu - 1),$$

and the distance $AB = c$, becomes

$$c = 2a\ \sin\ \delta = 2a\delta = 2a\ (\mu - 1)\ \varepsilon.$$

Therefore

$$x = \frac{(a+b)}{c} \cdot n\frac{\lambda}{2} = \frac{a+b}{2a\ \varepsilon\,(\mu - 1)} \cdot n\frac{\lambda}{2} \qquad (282)$$

The biprism has the advantage that the fringes are very bright and are readily obtained, while the adjustment for the two mirrors is exceedingly delicate, difficult to secure and troublesome to maintain.

198. Interference in Thin Films. Films of any transparent substance if sufficiently thin, exhibit brilliant colors when viewed by white light, or a series of dark and bright bands if examined by monochromatic light. Examples of such films are seen in soap bubbles, thin films of glass, thin films of oil upon water, or of oxide upon heated surfaces of polished metals. In order to comprehend the relation of these interference bands to other interference phenomena it is necessary to conceive the two interfering pencils as produced by reflection at the upper and lower surfaces of the film. Thus let $B\,C$ (Fig. 127), represent a uniform thin film of air enclosed between two plates of glass. When monochromatic light falls normally upon the film a part of the light is reflected at the upper surface B, in the direction BA, the other part penetrates the film, and part of this is reflected at C in the same direction CBA, while the remainder passes through the plate. In Fig. 127 the reflected parts are denoted by V and H. Both reflected components of the original beam are propagated in the direction BA, and the illumination

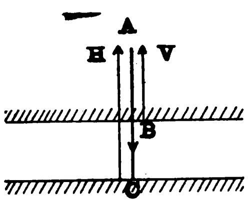


Fig. 127.

produced by H and V depends upon their phase difference, which *seems to be* the double thickness of the film, $2t$, expressed in wave lengths of the light under consideration. From our previous study of interference phenomena we should expect that according as this difference in path $2t$, amounts to 0, λ, 2λ, etc., or to $\lambda/2$, $3\lambda/2$, $5\lambda/2$, etc., that is, according as the difference in path amounts to an *even* or an *odd* number of half wave lengths, the film, to an eye looking down upon it along AC, should appear *bright* or *dark*.

Observation shows that exactly the opposite result is attained. When the film is made as thin as possible *it appears black*, thus showing that for a vanishingly small thickness t, the two pencils are in *opposite phase* instead of *coincidence*.

The reason for this is found in the opposite conditions of reflection undergone by the two pencils; the beam reflected at B is reflected *in* glass *against* air, while the beam reflected at C is reflected *in* air *against* glass. Theory shows that when reflection occurs *in* the rarer medium *against* the denser, the wave crest is reflected as a crest, while in the case of reflection *in the denser against the rarer* medium, a wave crest is reflected as a trough. This is usually expressed by saying that in the case of reflection *in air against glass*, a half wave length is lost. Fig. 128 shows the simultaneous positions of the ether particles in the two beams, where the incident wave is represented by the full line and the reflected waves by the dotted lines.

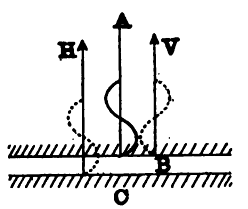


FIG. 128.

In a film of vanishingly small thickness therefore, the two wave trains as superimposed in opposite phases and the result is darkness. The total difference of phase is therefore that due to difference in path, $2t$, *plus* that due to difference in conditions of reflection $\lambda/2$. Hence the total retardation is $2t + \lambda/2$, and when *this quantity* amounts to an *even* number of half wave lengths the film is bright. When it is equal to an *odd* number the film is dark.

In the case of oblique illumination it is easily shown that the retardation due to the film is $2t \cos r$, where r is the angle of refraction in the film. Hence for an eye looking down upon a

film of uniform thickness, the path retardation increases with the obliquity. The film will therefore present a series of interference fringes in the form of circles concentric about the foot of the perpendicular from the eye upon the plane of the film, the circles being the loci of points of equal phase difference.

If the film be wedge-shaped instead of plane parallel, it will be crossed, when viewed by reflected light, by a system of dark bands which run parallel to the edge of the wedge, and mark the places at which the path retardation is 0, $\lambda/2$, $2\lambda/2$, $3\lambda/2$ . . etc. Since for extinction of the light the total retardation $2t + \lambda/2$, must equal an *odd number* of half wave lengths, we have *for the dark bands*

$$(2n + 1)\frac{\lambda}{2} = 2t + \frac{\lambda}{2} \tag{283}$$

where n is any integer, 0, 1, 2, 3, etc., and denotes the number of the dark band under consideration. The dark band corresponding to $n = o$, denotes optical contact of the two surfaces, or zero thickness of the film.

199. Interferomenters. Any device whereby two pencils of light may be made to interfere, and the resulting phenomena studied, is, properly speaking, an interferometer. Certain forms of the instrument present peculiar advantages, and merit special description. In the interferometer originally devised by Fizeau, and modified and improved by Abbé and Pulfrich, the two pencils are made to interfere after normal reflection upon the surfaces of two horizontal plates of glass enclosing a film of air, exactly as described in the preceding article, Figs. 127 and 128. The other parts of the instrument are for convenience of observing and measuring the fringe system.

The under plate of glass is actuated by a micrometer screw so that the thickness t, of the air film may be varied at will. The plates are slightly inclined to each other, thus giving the air film a wedge shape. The interference bands are then straight lines parallel to the edge of the wedge. On decreasing the thickness of the air film the bands move toward the thicker part of the wedge; on increasing the thickness they move in the reverse direction. Hence to measure the wave length of any colored light, the sys-

tem is illuminated with that particular light and the micrometer head turned until a definite number b, of dark bands have passed under a certain point on the plate. The amount s, by which the thickness of the film has been changed is determined from the micrometer head and the quantities b and s inserted in the general formula.

Thus for the dark band under the mark we have by (283)

$$(2n+1)\frac{\lambda}{2}=2t+\frac{\lambda}{2}\ (before)$$

also, since to pass from any dark band to the next higher one, we must introduce a difference of path of *one whole wave length*, we have, after passing over b dark bands,

$$[(2n+1)+2b]\frac{\lambda}{2}=2(t+s)+\frac{\lambda}{2}\quad(after)\qquad(284)$$

Whence

$$2b\cdot\frac{\lambda}{2}=2s$$

or

$$\lambda=\frac{2s}{b}\qquad(285)$$

In the Michelson interferometer, a beam of light from the source Q, (Fig. 129), falls at an angle of 45° upon the half-silvered face of a plane parallel glass plate A, where it is divided into two pencils, one of which is transmitted and passes to the mirror D, the other is reflected to the mirror C. These two mirrors are set so as to return the two pencils upon their paths to the point A, where the first is reflected and the second transmitted to E. A second plane parallel plate B, of the same thickness as A is inserted in the path of the reflected ray to make the paths traversed by the two pencils meeting at E equal, in case D and C are symetrically placed with respect to A.

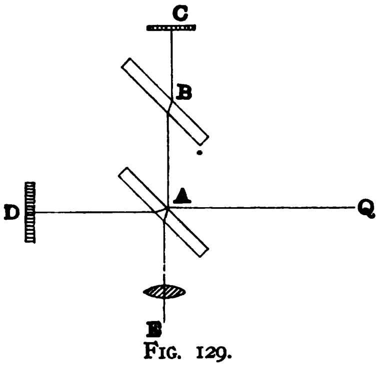


FIG. 129.

Now the transmitted ray has passed through the plate *A* three times and been reflected once *in glass against air.* The reflected ray has passed twice through *B*, once through *A*, and been reflected once *in air against glass.* When the two pencils have traversed equivalent paths they are in condition to interfere owing to the half wave-length difference in phase introduced by opposite conditions of reflection at *A*. One of the mirrors *C*, is movable in the direction *AC* by means of a micrometer screw. Wave lengths may be measured as in the Pulfrich-Abbé instrument.

The Michelson interferometer has the advantage of a wide separation of the two pencils of light, and it may consequently be applied to an almost endless variety of physical problems.

CHAPTER XXIII.

DIFFRACTION

200. Diffraction Through a Narrow Slit. If a strong beam of parallel rays be passed through a vertical slit into a darkened room and received upon a white screen some two or three meters distant, there will be seen a central band of white light, broader than the dimensions of the slit would justify from strictly rectilinear propagation, and on either side a series of colored fringes. It is apparent that through a narrow slit the light does not travel in straight lines even approximately, but bends around the edges, and spreads out in all directions, from all points of the opening as new centers of subsidiary waves. Such a phenomenon is called *diffraction,* and the fringes are called *diffraction fringes.* As has been pointed out already, diffraction is a species of interference between waves arising from different points on the *same wave front.*

Let *ab* (Fig. 130), represent a horizontal section through the vertical slit, and let *r* be a point on the screen *M N,* such that $ar = br$. Then the disturbances from all points in the slit will reach *r* in practically the same phase. They will therefore reënforce each other and *r* will be a bright point for all colors, and the central band will be *white.* If now a point *s* be taken near *r,* such that $as - bs = \lambda$, for some definite color, as the violet, then the waves setting out from *b* would reach *s* one whole period ahead of the waves from *a.* If *d* be a point midway between *a* and *b,* then a wave from *d* would arrive at *s* just one-half period later than the one from *b,* and hence these waves would annul each other. In like manner every elementary wave from a point in *da,* would be cancelled by a wave from a corresponding point in *bd.* The result at *s*

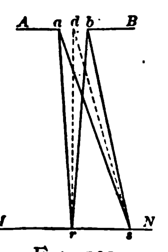


FIG. 130.

would therefore be zero *for violet light.* If on the other hand a point s_1 be chosen, such that

$$as_1 - bs_1 = 3\frac{\lambda}{2}$$

for violet light, then the slit ab can be divided into three parts, from *two of which the waves will interfere as before,* while the remaining third will produce violet illumination.

In general, if

$$as - bs = \pm n\frac{\lambda}{2} \quad (286)$$

for points on either side of r, we shall have a series of bright and dark bands, for monochromatic light. The bright bands will correspond to the points where n is *odd* and the dark bands to the points where n is *even.* For white light the fringe system is a series of rainbow-colored bands, violet on the inner, and red on the outer edge, thus showing again that violet has the shortest, and red the longest wave length of the spectral colors.

201. The Diffraction Grating. If instead of a single slit, a series of parallel, equidistant slits be ruled upon a piece of smoked glass, or better upon the opaque film of a photographic plate, then the colors of the diffraction fringes are much more lively, and the phenomena in strong sunlight are very beautiful. Such a ruled surface is called a *diffraction grating,* and the resulting spectra are called *diffraction* spectra.



FIG. 131.

Let M_1N_1, N_1M_2 (Fig. 131), represent the transparent and opaque parts respectively of a diffraction grating, and suppose the light from a collinator slit to come in a parallel beam from the left. Then along the direction indicated by the dotted line from M_2 we shall have a bright band, which may be focussed by a lens into an image of the slit.

Next suppose that for light of wave length λ, emerging in direction N_1D_1, M_2D_2, the difference in path of the waves from corresponding points in adjacent slits of the grating as M_1 and

M_2, should be one wave length, or $M_2D_2 = \lambda$; then it is clear that the light from the first slit will be in advance of the light from the second slit by *a whole wave length,* and ahead of that from the third slit by *two wave lengths* and so on. The light from all the slits will, therefore, agree in phase and the resultant illumination may be focussed by the lens into a *diffracted* image of the slit. This is called the *first diffracted image* of the slit, or a diffraction image of the *first order.*

If the direction of the light for the first spectrum make an angle θ_1, with the normal to the grating, and $M_1N_1 = a$, and $N_1M_2 = b$, then since the angle $D_2M_1M_2 = \theta_1$, we have

$$M_2D_2 = \lambda = (a + b) \sin \theta_1 \tag{287}$$

Similarly for an angle θ_2 such that the difference in path between corresponding points of adjacent slits is 2λ, we have for the second spectrum

$$2 \lambda = (a + b) \sin \theta_2$$

for the third spectrum

$$3 \lambda = (a + b) \sin \theta_3$$

and so on. From equation (287) we have

$$\sin \theta_1 = \frac{\lambda}{a + b} \tag{288}$$

which shows that the sine of the angle of diffraction is directly proportional to the wave length, hence in a diffraction spectrum the violet light is *deviated least* and the red light *most.* Moreover since the deviation of every color is directly proportional to the wave length, it follows that each color lies in its proper place and that in such a spectrum there can be no irrationality of dispersion.

Again, since sine θ_1 varies inversely as $(a + b)$, the *"grating constant,"* it is clear that the lengths of diffraction spectra from two different gratings are inversely proportional to the constants of the two gratings and hence can be compared at once.

It is also to be noted that for $\lambda = a + b$ the value of $\sin \theta_1$, becomes unity. This means that for waves of this length the angle of diffraction becomes 90°. Hence for measurements in the *infra red* end of the spectrum, gratings of large constant must be used.

Rutherford ruled gratings upon glass having 700 lines to the

millimeter. The magnificent gratings of Professor Rowland are ruled upon speculum metal and have in some cases as many as 1,700 lines to the millimeter.

202. Measurement of Wave-lengths. The diffraction grating affords the simplest method for the measurement of wave-lengths of light since, if the grating constant $(a+b)$ be known, the process is reduced to the measurement of a single angle, which can be made with great accuracy. A grating is mounted vertically upon the table of a spectrometer and so adjusted that the beam of parallel rays from the collimator strikes the grating surface normally. The slit is illuminated with monochromatic light, furnished by a Bunsen burner carying a tip of asbestos paper saturated with sodic nitrate. On turning the telescope so as to look directly into the collimator the direct, or central image of the slit should be seen sharply focussed, when the telescope and collimator are set for parallel rays. On either side of this central image will be seen the yellow diffracted images of the slit, of the first, second, third, and fourth order. Owing to the small fraction of the aperture in each slit which is effective in producing illumination, the intensity of the diffracted images falls off very rapidly. The vertical cross hair of the telescope is set upon the first diffracted image on each side of the central image, and the readings taken. One half the difference of these readings is θ_1. In the same way the values of θ_2, θ_3, and θ_4 are determined. Then

$$\lambda = (a+b)\sin\theta_1 = (a+b)\frac{\sin\theta_2}{2} = (a+b)\frac{\sin\theta_3}{3}, \text{ etc.}$$

Conversely, if light of a known wave-length be used, the grating constant, $(a+b)$, may be at once determined.

203. Three Kinds of Spectra. Through the investigations of Kirchhoff and Bunsen, 1856-60, the following fundamental facts were established concerning the three more important types of spectra.

(*a*) *Bright line spectra.* When light from an incandescent gas or vapor is examined by means of a prism, its spectrum is seen to consist of a number of bright lines, colored images of the slit, which are always the same for the same gas under the

same conditions of temperature and pressure. Thus the spectrum of sodium vapor at the temperature of the Bunsen burner, consists of a single pair of bright yellow lines corresponding to the Fraunhofer lines D_1 and D_2. The spectrum of lithium consists of a single line in the deep red. The light from hydrogen in a Geissler tube shows four well marked lines; one in the red and one in the blue corresponding to the Fraunhofer lines C and F, and two fainter lines in the violet.

Such a spectrum is called a *bright line spectrum* and its presence indicates to us that the source of light is a mass of gas or vapor under a pressure so low as to allow the gas molecules sufficient freedom of motion to execute whatever form of vibration they will. From the fact that the spectrum of a chemically pure substance in the gaseous form may contain *numerous bright lines*, we are driven to the conclusion that the molecule of such a gas may execute a number of different vibrations at the same time, just as a string or a plate may vibrate in a number of different modes, and produce a number of corresponding tones at the same time. Under this aspect of the case the characteristic bright lines in the spectrum of any gas at a given temperature may be regarded as representative of the *free vibrations which its molecules can execute at that temperature.*

(*b*) *Continuous Spectra.* When the light from an incandescent solid or liquid, or from a mass of incandescent gas *under high pressure* is analyzed, the spectrum is found to contain all colors from red to violet, and to show no discontinuities at any point. Such a spectrum is called a *continuous spectrum* and shows that the source is a mass of incandescent solid, liquid or gas at high pressure. The spectra from flames, molten metals, the filaments of incandescent lamps or from the carbon tips of an arc lamp are all continuous spectra.

In the case of such a luminous source, it is clear that the molecular motions due to incessant molecular collisions, must be extremely irregular and constantly interrupted. The molecules have practically no mean free path, and no time in which to execute their characteristic vibrations. The result is a confused, jangled mass of vibration of every possible frequency, which the eye interprets as light of all colors, *i. e., a continuous spectrum.*

(*c*) *Dark Line, or Absorption Spectra.* Absorption spectra

are produced when light from an incandescent solid is passed through a layer of some unequally transparent medium, and then analyzed. The spectrum is seen to be crossed by one or more dark lines or bands, indicating that in these regions the energy of the spectrum has been absorbed by the medium under investigation. Liquids are examined for absorption by placing them in tanks with parallel sides of plane parallel glass plates.

Many substances present characteristic absorption spectra. Thus a piece of cobalt glass absorbs all colors except a small strip in the red, and the blue end of the spectrum. The absorption spectrum of chlorophyl shows a dense black line in the red, while blood, even in very dilute solution, shows two characteristic bands in the green.

In the case of a glowing gas the absorption spectrum exhibits one or more dark lines sharply defined upon the continuous spectrum of the source. These lines are characteristic for the gas and correspond to certain bright lines emitted by the same gas. Thus at the temperature of the Bunsen burner, sodium vapor absorbs only the yellow rays belonging to the *D* lines.

The principle of absorption is merely the principle of resonance applied to the motion of ether particles. The light emitted by a vibrating atom of a heated gas may be considered as representative of the vibrations which it can execute. If those same vibrations fall upon the gaseous atoms they will covibrate or take up the vibratory motion, just as a tuning fork will respond to vibrations of its own frequency but to no others.

It is to be noted further that the glowing gas acts at the same time both as an absorbing and as an emitting layer. If light from a source at a temperature higher than that of the gas pass through the gas, then the gas molecules take up more energy than they give out, or light is absorbed by the gas. But if the gas be at the higher temperature, then the gas molecules give out more energy than they absorb, and light of that peculiar quality is added to the light of the source. In the first case the lines are darker the greater the degree of absorption, *i. e.,* the greater the difference of temperature. In the second case the line shows as a bright line on the continuous spectrum. For equality of temperature between the sources the line vanishes.

204. Kirchhoff's Law. From a study of the absorptive powers of glowing gases, Kirchhoff deduced a law connecting the radiation and absorption of bodies, which may be stated as follows: If I be the intensity of the monochromatic light which falls upon a body at a given temperature, and I' be the part of the light absorbed by that body, then Kirchhoff terms the ratio

$$\frac{I'}{I}=A \tag{289}$$

the absorptive power of that body for that particular color. Also if E, the intensity of the light of the same color which the body *emits* at the same temperature be called the *emissive power* of the body, then Kirchhoff's law states, that *the ratio between the absorptive power and the emissive power is the same for all bodies at the same temperature,* and that *the value of this ratio depends only on the temperature* and the *wave-length.* An *absolutely black body* is defined as one which absorbs all the light which falls upon it, and does not reflect or transmit any. For such a body, therefore, the absorptive power I'/I is unity for all colors and for all temperatures. Now according to Kirchhoff's law we have the ratio E/A for one body, the same as that for another, hence

$$\frac{E}{A}=\frac{e}{1} \tag{290}$$

where e is the emissive power of the black body for the given color and temperature. From this it follows at once that the ratio E/A, for *any body* is equal to the emissive power of a *black body* for the same color and temperature. This means that if a vapor, as sodium vapor, at a given temperature, emit light of a certain color more abundantly than other colors, it will *absorb that same color* more abundantly, since

$$\frac{E}{A}=e, \text{ a constant.} \tag{291}$$

If we write the law in the form $E = eA$ and remember that A can never be zero for any body, we reach the important conclusion, that as soon as a black body reaches a temperature where it begins to radiate a given color, *as red light,* then *any other body begins to radiate red light at the same temperature,* or *all bodies become red hot at the same temperature.* This tempera-

ture is about 525°C. As the temperature rises, shorter wave-lengths are radiated in addition to the red. At 1000° the body becomes yellow, and at 1200°, white.

205. Spectrum Analysis. Since the character of the light emitted by an incandescent gas depends first of all upon the vibrations of its constituent atoms, it follows that a study of the light emitted by a glowing gas gives us direct testimony concerning its chemical composition. Hence, if the bright line spectrum of any substance be once definitely known, then whenever this spectrum presents itself we may conclude at once that the given substance is present in the source of light, whether that source be a Geissler tube in the laboratory or a fixed star in the depths of space. This is the method of spectrum analysis. A minute quantity of a salt is introduced into the colorless flame of a Bunsen burner and the light examined by the spectroscope. The method is most useful in the detection of the metallic constituents of salts, since at the temperatures necessary to vaporize a salt and tinge the flame, the spectrum is generally independent of the acid constituent.

The method is characterized by its ease and rapidity, and especially by its exceeding sensitiveness. In the flame of the Bunsen burner, 1/14,000,000 of a milligram of sodium is sufficient to show the characteristic sodium lines, while in the spark of an induction coil, 1/80,000,000 of a milligram of lithium may be detected. On account of the extreme sensibility of the method it has led to the discovery of numerous new elements, which have been present in minute quantities as impurities in the substances under examination, and revealed themselves through characteristic new lines in the spectrum. Among the elements so discovered may be mentioned caesium, rubidium, thallium, indium and gallium.

Spectrum analysis gives at once the explanation of the Fraunhofer lines in the solar spectrum, and enables us through comparison with bright line spectra from known sources to prove the presence of many chemical elements in the sun. Thus the two *D* lines of the solar spectrum coincide exactly with the two yellow lines of the spectrum of sodium, and Kirchhoff concluded that there must be sodium vapor in the sun's atmosphere. By

means of the Fraunhofer lines Rowland has definitely proven the presence of thirty-six chemical elements in the sun.

Again, it should be noted that while the spectrum of any substance is characteristic of that substance, and furnishes a reliable criterion for conclusions concerning the constitution of its molecule, yet the same substance may exhibit different spectra for different conditions of pressure and temperature. It seems reasonable to suppose that a complex molecule is capable of more varied forms of vibration than a simpler one, and hence it seems likely that a complicated spectrum corresponds to a complicated molecular structure, and a simple spectrum to a simple molecular structure. Experiment seems to show that *each compound that can exist at the temperature at which light is emitted, has its own spectrum.* As the temperature of a solid rises, the spectrum changes correspondingly.

From the first appearance of color, the spectrum grows to the completed, *continuous* spectrum. For a slightly higher temperature, but one at which the compound molecule can still exist, the spectrum is marked by the appearance of bright parts, which are not yet sharp lines, but rather broad bands, set off by bright lines which shade off into darkness on one side. Such a spectrum is called a *band spectrum* or a *fluted spectrum,* since it has the fluted appearance of a Greek column. Such spectra correspond to relatively low temperatures and complicated molecular structure, usually that of a chemical compound.

For still higher temperatures the compound molecule breaks up into its constituent atoms, and the *bright line* spectrum appears. This as we have seen, corresponds to a highly heated gas under low pressure.

Again, the spectrum of an element may contain but a few bright lines that seem to be arranged in some apparently definite, rythmical order, as in the spectrum of lithium or it may contain a thousand lines arranged in apparently the greatest confusion as in the case of iron. These lines may also occur singly as in lithium, or in pairs as in sodium, or in triplets as in magnesium. These groups of single lines, or pairs or triplets recur at regular intervals, the intervals growing shorter as we approach the violet end of the spectrum. The analogy between these rythmically recurring groups of lines and the overtones produced by a sound-

ing body is certainly very striking. Through the investigations of Kayser and Runge, certain harmonic relations between the vibration frequencies of the spectral lines of many of the elements have been established, but these relations are by no means simple, nor have they as yet been shown to exist in the case of all the elements.

206. The Visible and the Invisible Spectrum. It has been seen that many substances exhibit marked absorption for light of definite wave-lengths. Investigation has proven that this absorption is manifested even by transparent substances for longer or shorter wave-lengths. By means of the photographic plate it is possible to follow the spectrum far beyond the point at which it ceases to affect the eye, and to map out the dark lines in the invisible solar spectrum, as Fraunhofer did in the visible part. To do this, however, it is necessary to avoid the absorption of prisms and lenses. For this purpose a spectrum is produced by a diffraction grating and focussed by means of concave mirrors. In the Rowland's concave gratings, the focussing is done by the grating itself.

The study of such spectra shows that the luminous spectrum which we can perceive by the eye is only about one-thirtieth of the spectrum that has now been explored. The only physical difference between visible and invisible light is one of wave-length. The eye like the ear is limited in its range of sensibility and responds only to those vibrations whose wave-lengths lie between 0.000812 and 0.000330 mm. For wave-lengths shorter than these the photographic plate must be used. This region is termed the *ultra violet* end of the spectrum.

In the region of longer waves, called the *infra red*, recourse is had to the heating effect of the radiant energy, and the spectrum is explored by means of an eye-piece containing a series of thermal couples. This is connected to a sensitive galvanometer and shows the presence of bright or dark lines as *hot* or *cold* lines, by the deflection of the galvanometer. The same instrument may be used to detect the presence of maxima and minima in a system of electric waves, and consequently enables us to explore the spectrum of electric radiation as well. The conclusion to which we are driven by these facts is, that *radiant energy, i. e., energy*

transmitted as undulations in the ether, is essentially one in kind, whether it manifest itself in the form of electric waves, or heat waves, which affect the thermal couple, or light waves which affect the eye, or ultra violet waves which affect the photographic plate. The only essential difference is that of wave-length.

Again, no one substance is equally transparent for all wave-lengths. Fluor spar alone transmits the shortest wave-lengths; quartz transmits a large part of the ultra violet. Glass is transparent for the visible part of the spectrum but absorbs the infra red. Rock salt is very transparent for all but the longest heat waves, while for the longer electric waves an oak plank or a brick wall is fairly transparent.

The following table of wave-lengths gives some idea of the range of the spectrum thus far explored, and of the relatively small part occupied by the visible spectrum.

WAVE LENGTHS.

Kind of Light.	λ in mm.
Limit of the photographic rays in vacuum	0.000100
Limit of the photographic rays in air	0.000185
Limit of visible light in the blue	0.000330
Blue hydrogen line	0.000486
Yellow sodium line	0.000589
Red hydrogen line	0.000656
Limit of visible light in the red	0.000812
Longest heat waves as yet detected	0.06
Shortest electric waves	~~6.~~ 3. (Bose)

CHAPTER XXIV

RESOLVING POWER OF OPTICAL INSTRUMENTS

207. Resolving Power of the Telescope. The performance of every form of optical instrument reaches a limit imposed by the nature of light itself. If a plate ruled with fine parallel equidistant lines, be examined, either by the unaided eye, or by means of a telescope or a microscope, there will in each case be found a limiting distance s, between the lines, below which they are no longer seen as separate and distinct lines. This limiting value of s, is termed *the limit of the resolving power* of the instrument in question, or more briefly, *the resolving power.* The principles involved in the determination of this limiting value may best be explained from the telescope.

Suppose ES (Fig. 132), to be a narow slit and O its middle point. Let p be any point on a line normal to ES through O, and at a distance from the slit, then p is a bright point for light of any color entering the slit. Next let p' be a point on a perpendicular

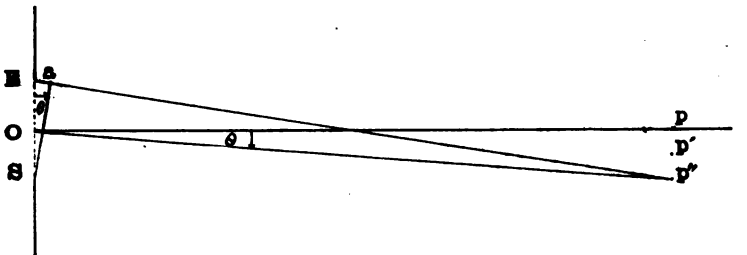


FIG. 132.

to Op through p, such that $Ep' - Sp' = \lambda/2$ for light of some definite wave-length. Then waves starting from the points E and S will reach p' in opposite phase, and so annul each other. But it is *only for the extreme edges* E and S that this is true, hence there will be *some* illumination at p'. But if we take a point p'', such $Ep'' - Sp'' = \lambda$, then p'' is a dark point for light of wave-length λ. (See Article 200.)

We should therefore have a series of bright and dark bands, parallel to $E\,S$, and normal to the plane of the paper, of which p'' marks the *first dark band.* If now the slit $E\,S$, be regarded as the diameter D, of a telescope objective, we have a series of bright and dark rings concentric about the bright center p. This bright center may be regarded as the image of a point source of light of wave-length λ. Calculation shows the radius of the first dark ring to be slightly larger than the value derived for the first dark band. Thus the radius of the first dark ring is $1.2\ pp''$, or the diameter is $2.4\ pp''$.

Let $E\,S = D$, $Op = F$, the focal length of the objective, $d = 2.4\ pp''$, the linear diameter of the image of the point source, and $E\,a = \lambda$. About p'' as a center, with $p''S$ as radius describe the short arc Sa. Then the angles ESa and pOp'' are equal, and equating values of sine and tangent we have

$$\frac{Ea}{ES} = \frac{pp''}{op}$$

or

$$\frac{\lambda}{pp''} = \frac{D}{F} \qquad (292)$$

whence $2.4\,pp'' = 2.4\,\lambda\,F/D = d$, the linear diameter of the image of the artificial star. The angle subtended by this image at C, the center of the objective is d/F, which (Fig. 133), is readily seen to be the same as that subtended by the object at the same point. Then $d/F = 2.4\lambda/D =$ *the angular diameter* of the star image. If two star images are to be seen as separate discs, then their angular separation must be at least

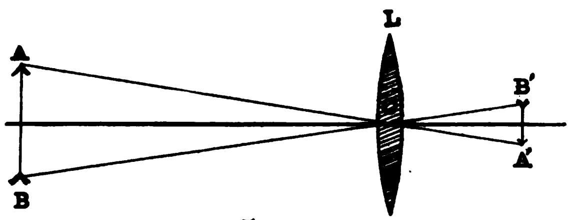


FIG. 133.

$$\frac{d}{F} = \frac{2.4\,\lambda}{D} \qquad (293)$$

If we set $\lambda = 0.00056$ mm., and $D =$ one inch $= 2.54$ cm., then the angular diameter of the star disc is 10.92″, or the images of two stars which subtend an angle of 10.92″ would appear in the telescope with an objective one inch in diameter, with their discs just touching, if the light could be traced out to the edges of the discs. On account of the faint light of the stars, however, the extreme edges of the discs are invisible, and under most favorable circumstances two stars can be separated, which are a little less than half this distance apart, or *the limit of the resolving power of a telescope whose objective is 1 inch in diameter is about 5″.* Hence to find the resolving power of any telescope divide 5″ by the diameter of the objective in inches.

208. Resolving Power of the Eye. It was shown in the last article that the resolving power of a telescope depends simply upon the wave length of light employed, and upon the diameter of the objective, or two objects to be resolved must subtend an angle

$$\phi > \frac{2.4\,\lambda}{D} \qquad (294)$$

In the case of the eye, the crystalline lens has a refractive index of 1.4, and hence the wave length λ in air, becomes $\lambda/1.4$ in the lens of the eye. The diameter of the pupil corresponds to the diameter of the objective of the telescope, and putting this diameter equal to 4 mm., we have as the resolving power of the eye

$$\phi = \frac{2.4}{4} \cdot \frac{\lambda}{1.4} = 49.5'' \qquad (295)$$

The actual limit is about one minute. This means that a normal eye can see two lines separated, whose distance apart subtends at the eye *an angle of one minute of arc.*

It is interesting to note that the "rods and cones" or light sensitive elements of the eye, subtend the same angle, one minute, at the nodal point of the eye. From this it appears that whether we consider the resolving power of the eye as depending upon the smallest distance between two nerve endings capable of receiving separate stimuli, or whether the eye be regarded as a simple lens, the theoretical resolving power comes out the same in either case. The resolving power of the eye may be readily determined by the following experiment.

Draw a series of equidistant lines upon a piece of white paper, making the lines and spaces of about the same width, and determine the distance from the eye at which the lines under bright light can just be seen resolved. The angle subtended at the eye by the distance between two lines will give the resolving power for the eye in question.

209. Resolving Power of the Microscope. It has been pointed out that the magnification of a simple lens soon reaches a limit owing to the short working distance and the minute dimensions of the lens. A more serious difficulty is found in the fact that as the lens grows smaller the dimensions of the opening are no longer very great as compared to the wave length of light. The image of a point source is therefore a diffraction disc of definite radius and this radius increases as the diameter of the lens decreases. It can be shown experimentally that for a lens whose diameter is less than one one-thirtieth of an inch the confusion arising from the increased diffraction is very great.

The *angular aperture* $2a$, is the angle included between the extreme rays, which can pass through the microscope objective, from a point distinctly seen on the axis of the instrument. If μ be the index of the medium *from which* the light enters the objective, then $\mu \sin a = N$, is called the *numerical aperture* of the objective. In air $\mu = 1$ and the maximum value for N, is also 1. Since this would denote an angular aperture of 180°, it is obvious that the greatest diameter D, that can be used in a simple lens is twice the focal length F, or

$$N = \frac{D}{2F} \tag{296}$$

where N has its maximum value of 1.

By placing upon the cover glass, a drop of some fluid in which the objective may be immersed, the factor μ may be varied at pleasure and the aperture N may be increased to 1.4 or even more. Such a lens system is termed an *immersion system.* If a liquid be used whose index is the same as that of the objective, it is termed *homogeneous immersion.* If a grating of constant. $(a + b) = d$, be viewed by a microscope in direct light, then the diffraction pattern in the image will resemble the structure of the object (grating), only when all the rays diffracted by the

object of sufficient intensity to produce appreciable effects in the focal plane of the objective, are received by the objective. Hence the resolving power depends directly upon the numerical aperture of the objective. When the grating is viewed by direct light, therefore, the first maximum from the center of the field, lies in the direction $\sin \theta = \lambda/d$. Hence if the grating is to be seen resolved, that is, if the lines are to be seen separated in the image, the objective must receive rays whose inclination is at least equal to θ, where

$$\sin \theta = \frac{\lambda}{a} \qquad (297)$$

In an immersion system, the wave length λ' in the fluid, is equal to λ/μ, where μ is the refractive index of the immersion fluid, and λ is the wave length in air. In this case

$$\mu \sin \theta = \frac{\lambda}{d} \qquad (298)$$

for resolving a grating of constant d. But since $\mu \sin \alpha$ is the numerical aperture N, of the objective, then to resolve the grating, $N = \mu \sin \alpha$ must equal $\mu \sin \theta$, or

$$N = \frac{\lambda}{d}$$

Hence the smallest distance d, which can be resolved by a microscope of numerical aperture N, in direct illumination is

$$d = \frac{\lambda}{N} \qquad (299)$$

In the case of oblique illumination this may be reduced one-half under *the most favorable conditions,* or

$$d = \frac{\lambda}{2N} \qquad (300)$$

Since the resolving power *increases as d decreases it is clear that the resolving power of the microscope varies inversely as the wave length of light used, and directly as the numerical aperture of the objective employed.*

Letting $\lambda = 1/50000$ of an inch or 0.000508 mm., for greenish blue light, and putting $N = 1$, we have for *oblique illumination, in air, under most favorable conditions,*

$$d = \lambda/2N = 0.00001 \text{ inch or } 0.000254 \text{ mm.}$$

This means that under the above conditions a grating having 100000 lines to the inch could be seen resolved. In the case of homogeneous immersion this limit may be extended to 135000.

210. Resolving Power of a Grating. If light fall upon a diffraction grating of constant $(a+b) = d$, at an angle of incidence α, and the transmitted light be diffracted at an angle β, then it may be shown that for the points of maximum illumination the maximum phase difference between wave systems from corresponding points of adjacent slits of the grating is

$$d\,(\sin\alpha + \sin\beta) = m\lambda \qquad (301)$$

This is the general form of the equation for the grating and applies equally well to reflection or transmission gratings.

In order to investigate the resolving power of the grating it is necessary to inquire into the conditions necessary to separate the mth spectral image of wave length λ, from the mth image of wave length $\lambda + d\lambda$, where $d\lambda$ is a very small fraction of a wave length. The solution given by Lord Rayleigh is as follows: In a grating of n lines, the mth spectrum lies in such a direction that the phase difference between corresponding points of adjacent slits is $m\lambda$, and between wave systems from the extreme slits, it is $m n \lambda$. The nearest points of minimum intensity on either side of the mth maximum correspond to phase differences between the extreme slits of $m n \lambda \pm \lambda$. If now the mth maximum for wave length $\lambda + d\lambda$, fall in the position of phase difference $m n \lambda + \lambda$, then the two lines are seen sharply separated, and we may equate length of paths for this point, or

$$(m n + 1)\lambda = m n\,(\lambda + d\lambda) \qquad (302)$$

whence

$$\frac{\lambda}{d\lambda} = m n = r$$

the resolving power of the grating. The quantity $r = \lambda/d\lambda$ indicates by what fraction of a wave length two lines must differ in order to be completely separated. For example, the two sodium lines D_1 and D_2, having wave lengths 5896 and 5890 $\times$ 10^{-7} mm., differ by 6 units, or the ratio $\lambda/d\lambda$ may be said to have, in round numbers, the value 6000/6, or 1000. A grating which will separate the D lines must therefore have a resolving power of at least

1000. Also, since $r = \lambda/d\lambda = m\,n$, it is plain that if the sodium lines are to be seen separated in the *first* spectrum the grating must have at least 1000 lines, while for the second spectrum 500 would suffice.

Again, if the general equation for the grating be multiplied by n, the number of lines on the grating, we have

$$m\,n\,\lambda = n\,d\,(\sin\alpha + \sin\beta)$$

But $m\,n = r$, the resolving power of the grating, and $n\,d = b$, the breadth of the grating, hence we may write

$$r \doteq \frac{b}{\lambda}(\sin\alpha + \sin\beta) \qquad (303)$$

This equation shows that the resolving power is a maximum for α and β each equal to 90°, or

$$r_{max} = \frac{2b}{\lambda}$$

This value, however, cannot be attained since it corresponds to an infinitely small bundle of rays. If either α or β be made zero, the other angle may amount to 60°. By the use of the Littrow autocollimation principle (Art. 160), in which telescope and collimator are combined, it is possible to make $\alpha = \beta = 45°$ to 50°; so that practically the maximum resolving power may be set down as

$$r = \frac{7b}{8\lambda} \text{ to } r = \frac{3b}{2\lambda}$$

From this it follows that for the largest and best gratings now made, in which $b = 13.2$ cm., the resolving power for $\lambda = 5500 \times 10^{-7}$ mm., is about 375000, if the Littrow method be adopted. Other arrangements give much less. In the usual arrangement of the Rowland concave grating the resolving power probably does not exceed 100000.

CHAPTER XXV.

POLARIZATION

211. Polarization of Light. Throughout the various optical phenomena thus far studied, there has been no indication as to the nature of the ether vibrations, which have been assigned as their cause. If, as in sound, the direction of vibration be *in the line of propagation*, in other words, if the vibration be longitudinal, then there will be nothing to distinguish the beam of light when viewed from one side, from its aspect when viewed from another. If a guitar string be plucked or a violin string be bowed so as to cause it to vibrate horizontally, then the entire excursions of its vibrating parts are confined to that plane and the motion is linear and simple harmonic in that plane. A card having cut in it, a narrow slit, slightly wider than the diameter of the string and several centimeters long, may be passed over the string *with the slit horizontal,* without disturbing the motion of the string in any way. If, however, the card be rotated in its own plane through 90°, so that the slit may stand vertical, the horizontal vibration is at once extinguished, although a vertical vibration would be rendered possible. It is clear that the card would have no influence upon the *longitudinal vibration* of the string in any position.

The transverse vibrations of a string are therefore such as to enable us to distinguish *its sides,* or to give to the string a *two-sidedness or polarity.* If now it could be shown that a beam of light behaves in a similar way, it would indicate the presence of transverse vibrations. If a beam of ordinary light be allowed to fall normally upon a plate of tourmaline cut parallel to the axis of the crystal, the light which emerges will be found to possess the two sided property of the vibrating string. If we allow the

light from one plate of tourmaline to fall normally upon a second similar plate, we shall find that it passes freely through the second when the two plates are parallel as at AB (Fig. 134). If, however, the second plate be rotated about the beam as axis, as $A'B'$, the emergent light gradually diminishes in intensity and is entirely extinguished when the two plates stand at right angles to each other, as at $A''B''$. If the second plate be rotated still further, the light again increases and when the plates are again parallel, it reaches its full intensity, to be extinguished again when the plates stand at right angles to each other. It thus appears that when the plates are crossed, the light from the first plate is stopped by the second, just as the vibrations of the string were stopped by the slit. The light has thus been changed in its nature so as to exhibit a *two-sidedness* or *polarity* in one plane and is therefore said to be *plane-polarized.* We also conclude that the vibrations of the ether particles are transverse to the line of propagation.

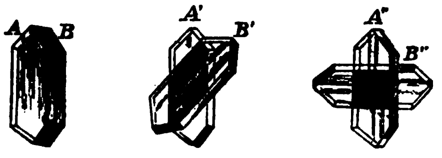


FIG. 134.

Again, since the light emerges freely from the first plate of tourmaline, no matter how it be rotated about the beam as axis, we conclude that in ordinary light, the transverse vibrations occur in all possible planes through the axis of propagation (Fig. 135*a*); whereas, the light transmitted by the first plate is due to the vibrations parallel to the longer axis of the plate (Fig. 135*b*). For this reason, ordinary light is considered to be made up of a mixture of light polarized in all possible planes, due to the rapid rotation of the plane of polarization about the line of propagation as axis.

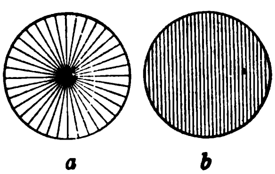


FIG. 135.

212. Polarization by Reflection. If a plate of unsilvered glass A (Fig. 136), blackened upon its rear surface, be placed

in a beam of ordinary light at an angle of incidence of about 57°, the light reflected from such a mirror, will be found to be *plane polarized*. This may be demonstrated by testing the beam by means of a plate of tourmaline, or by receiving the reflected beam upon a second similar mirror (Fig. 136), whose plane of incidence coincides with that of the first. The light in this position is freely reflected from the mirror B. If now the mirror B, be rotated about the beam AB as axis, the light reflected from the second mirror, will gradually diminish in intensity, until when the planes of incidence of the two mirrors are at right angles to each other, it vanishes entirely, $A'B'$ (Fig. 136*b*), as in the case of the tourmaline plates, and reappears again in its original intensity when the mirror is rotated through another 90°.

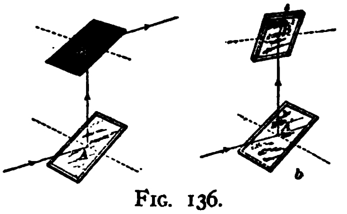


FIG. 136.

It should be noted that in the case of the crossed mirrors. the light reflected from B, is not zero except for a particular angle of incidence for both mirrors. This angle is called the *angle of polarization* and for glass it is between 55° and 57°. Other transparent substances may be used as mirrors, and for each substance there has been found an angle of incidence depending upon the substance, which gives a *maximum* of polarization, but, according to Jamin, only a few substances of refractive index near 1.46 polarize light completely by reflection.

If we suppose the beam $A\,B$, to be of unknown origin, then it may be *analyzed,* that is, its condition of polarization may be examined by means of the second mirror. If on rotating the mirror B about the beam as axis, the reflected light show no change in intensity, we conclude the beam AB is one of ordinary light. If, however, for certain positions of the mirror B, the light vanishes, it is *plane polarized* and the *plane of incidence in which it is reflected most copiously from the second mirror, is called the plane of polarization.* According to the theory of Fresnel, the vibrations of plane polarized light are perpendicular to the plane of polarization. Thus the direction of the vibration in

light polarized by reflection is normal to the plane of incidence, that is, *it is parallel to the surface of the mirror.*

213. Brewster's Law. It has already been stated that the angle of polarization differs for different substances. In 1811, Sir David Brewster discovered the remarkable fact, that when light falls upon a transparent substance, at the polarizing angle, the reflected and refracted beams *are at right angles to each other.* Thus, if EO (Fig. 137), be the incident ray, OR the reflected ray, and OB the refracted ray, then by Brewster's law the angle $BOR = 90°$. This law may be put in another form. Since the angle between the reflected and the refracted rays is equal to 90° ; then

$$i + r = 90° \text{ or } \cos i = \cos (90 - r) = \sin r,$$

therefore

$$\frac{\sin i}{\sin r} = \frac{\sin i}{\cos i} = \mu$$

or

$$\mu = \tan i. \qquad (304)$$

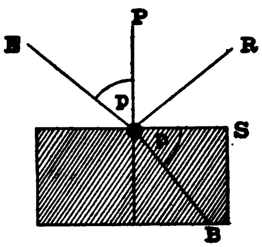


FIG. 137.

Now since μ is greater than unity we learn from Brewster's law that the polarizing angle is always greater than 45° and further, that if the index of refraction for a transparent substance be known, the polarizing angle can at once be deduced. Brewster's law has been verified by Seebeck for a number of refractive media.

214. Polarization by Refraction. If a beam or ordinary light fall upon a thin transparent glass plate at an angle of about 55° to 57°, a part of this light is reflected, and by this reflection polarized, the plane of polarization being *in* the plane of incidence. The other part of the light is transmitted, and if examined will be found to show traces of polarization in a plane *at right angles* to the plane of incidence. If the light emerging from the first plate be passed through a second parallel and similar plate, the amount of polarized light in the emergent beam is increased, and after passing through some eight or ten such parallel plates the transmitted light is found to be completely polarized. Such an

arrangement is called a *pile of plates* and may be used either as a polarizer or as an analyzer in optical apparatus.

The pile of plates may also be used to replace one or both the mirrors shown in Fig. 136. When the plates are used to replace mirror *B*, (Fig. 136), and the planes of mirror *A* and the plates are made parallel, the light reflected from *A* is also reflected from the first plate of the pile as it would have been from mirror *B*. If, however, the plates be rotated into the position of *B′* (Fig. 136), the light is no longer reflected by the plates but is wholly transmitted. Finally, if two piles of plates be used they behave toward each other exactly as two mirrors.

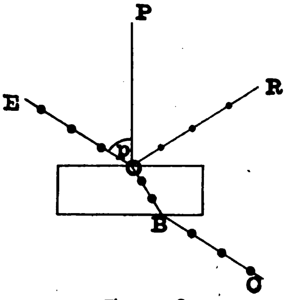


FIG. 138.

The action of the plates of glass may be understood from the following considerations: Let a beam of light fall upon a glass plate at the polarizing angle. If the light is already polarized in the plane of incidence (Fig. 138), then about 1/7 of the incident light is reflected along *OR*, the rest penetrates the plate. The vibration of the ether particles parallel to the surface of the plate are represented by the dots upon the path of the ray. If the incident light be polarized at right angles to the plane of incidence, (Fig. 139), then the entire beam penetrates the plate and no light is reflected along *OR*. In this case the vibrations in the plane of incidence are indicated as shown in Fig. 139.

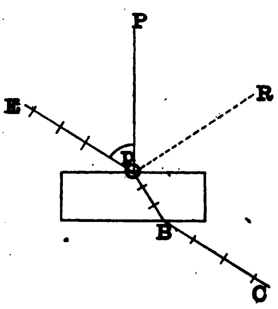


FIG. 139.

215. Double Refraction. When a ray of light falls upon a transparent isotropic substance it is refracted along a single direction, and the refracted ray obeys the law of refraction. When, however, a ray of light falls upon the surface of any transparent crystal other than one belonging to the regular sys-

tem, it is in general divided into *two refracted rays,* one of which obeys the law of refraction and is called the *ordinary ray,* while the other follows a law of refraction altogether different from that of isotropic bodies, and is called the *extraordinary* ray. This phenomenon is called *double refraction.* It is exhibited by many animal and vegetable substances, and by glass, glue, gelatine and similar substances when under stress. Double refraction is very readily observed in Iceland spar, (crystallized $CaCO_3$), in which it was first observed in 1669, by Erasmus Bartholinus.

Iceland spar belongs to the hexagonal system of crystals and splits readily in planes corresponding to the three faces of a rhombohedron. Two of the solid angles which lie diametrically opposite, are bounded by three equal, obtuse angles of 101° 53′, while each of the remaining six are bounded by one obtuse and two acute angles. A line $a\,a'$ (Fig. 140), making equal angles with the three faces forming the obtuse solid angles is called the *optic axis* of the crystal.

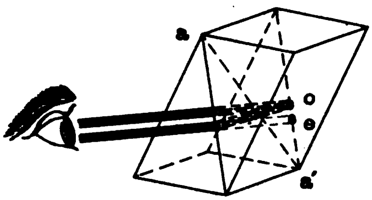


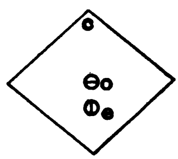
FIG. 140.

In the case of crystals which possess a principal axis of symmetry, a plane laid through this axis and including the normal to the surface, or any plane parallel to it is called a principal section. Consequently the plane which can be passed through the shorter diagonal of the rhombohedral surface and the axis of the rhombohedron is also a principal section. The rule for the double refraction in Iceland spar may be stated as follows: A ray of light incident normally upon a rhombohedral surface of Iceland spar is separated into two rays, one of which vibrates at *right angles* to the principal section and is not deviated, (ordinary ray), while the other vibrates *in* the principal section, and is deviated *in* the principal section, *away* from the end of the axis $a\,a'$ (Fig. 140), in the face *toward which* it is going, (extraordinary ray).

This may be illustrated by placing a rhomb of spar over a small black dot on white paper, and looking at the dot along the normal to the upper face of the cystal, (Fig. 140). The eye will perceive two images, one in the continuation of the normal to the

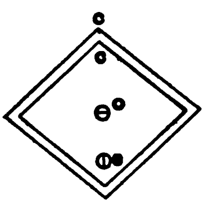
horizontal faces (ordinary image), and the other *in the principal section,* and *displaced from* the upper end of the principal axis, (extraordinary image). On rotating the crystal over the dot the extraordinary image rotates about the ordinary image but keeps its position relative to this image and the end of the axis *a* of the crystal.

216. Polarization by Double Refraction. If a ray of light be admitted through a small hole in a black card and a rhomb of Iceland spar be placed over it, the eye will perceive two rays emerging from the upper surface. These tested either with a polarizing mirror, or a pile of plates, will show the following peculiarities:


FIG. 141.

(*a*) The ordinary ray *o* (Fig. 141), will be found to be polarized *in the principal section of the crystal, i. e.,* it swings at *right angles to the principal section,* while the extraordinary ray is polarized at right angles to the principal section, or its vibrations are *in the principal section.* Fig. 141 represents the upper face of the rhomb in Fig. 140, and *c* denotes the end of the axis in the upper surface, designated by *a* in Fig. 140.

(*b*) If a second rhomb of equal thickness be similarly placed upon the first, that is, with the end of the optic axis in the upper face at *c* (Fig. 142), in each case, the same two images *o* and *e* (Fig. 142), appear as before, with similar polarization, but the separation of the two rays is now twice as great as before. This is easily explained since the two rays emerge from the crystal as parallel rays and enter the second crystal in the same relative position in which they left the first. Hence the ordinary ray traverses the second crystal as ordinary and the extraordinary as extraordinary. The ordinary ray penetrates the crystal normally and hence suffers no deviation while the extraordinary ray suffers the same deviation, in the second as in the first, and since the plates are of equal thickness the separation of the rays is twice as great for a rhomb of double the thickness.


FIG. 142.

(*c*) Next let the upper rhomb be rotated clock wise upon the lower, through an angle of about 30°. There will now appear *four* images instead of *two* in the positions shown in Fig. 143. Upon examination it will be seen that the ordinary ray O in the first rhomb has been split into two by the second, producing an ordinary O_o and an extraordinary E_o, while the extraordinary in the first rhomb is likewise doubly refracted in the second producing an ordinary O_o and on extraordinary E_o.

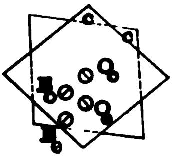


FIG. 143.

The intensities of the four rays increase and diminish in pairs. Thus the two rays O_e and E_o are at first faint and gradually increase while the pair O_o and E_e diminish in brightness and when the crystal has been rotated through 90°, vanish entirely leaving O_e and E_e, (Fig. 144). For a rotation of 45° all rays possess equal intensity. After passing 90° the pair O_o and E_e reappear, and increase in brightness, the other pair diminishing to zero at 180°. At this point the two remaining rays O_o and E_e coincide *for two rhombs of equal thickness* and we have *a single image caused by two beams of light polarized at right angles to each other,* (Fig. 145).

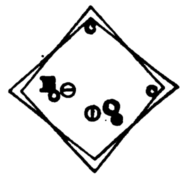


FIG. 144.

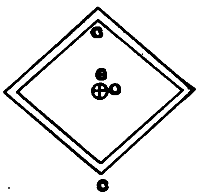


FIG. 145.

The paths of the rays through the two rhombs of equal thickness is shown in Fig. 146. The variation in the relative intensities of the for rays is readily understood from Fig. 147. Let JA and JB represent the directions of vibrations of the ordinary and extraordinary ray in the first rhomb, and JF and JD the corresponding directions of vibration imposed upon the two rays of light upon entering the second rhomb which has been rotated upon the first through

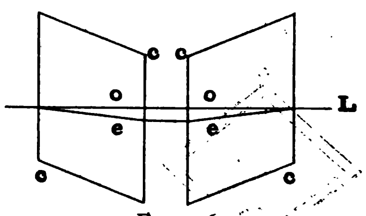


FIG. 146.

an angle OJD. Let JA and JB represent the intensities of the two rays on emerging from the first rhomb. By projecting JA and JB upon each of the new axes in turn, we have the four amplitudes of vibration,

$$JD = O_o = JF = E_e$$

and

$$JG = O_e = JC = E_o$$

whence the subsequent variations may be readily deduced.

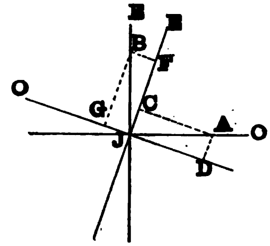


FIG. 147.

217. Indices of Refraction in Iceland Spar. Since a ray of light in passing through a rhomb of Iceland spar is split up into two rays, which are differently refracted it follows that the crystal must have two indices of refraction. In determining the refractive indices of any uniaxial crystal it is convenient to employ the form of prism described in Article 160, and arrange the apparatus so that the incident light is normal to the first surface of the prism. In each case the direction of the optic axis in the prism is indicated by fine lines.

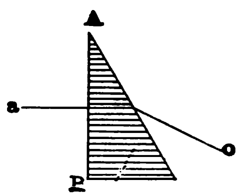


FIG. 148.

Three cases will be considered:

(*a*) *The optic axis of the crystal is parallel to the incident ray,* (Fig. 148). In this case the ray traverses the crystal parallel to the optic axis and no double refraction results. The refractive index for sodium light is that for the ordinary ray, $\mu_o = 1.6585$.

(*b*) *The optic axis is parallel to the refracting edge of the prism,* (Fig. 149). Here the plane of incidence is normal to the axis, and consequently two rays emerge from the second side of the prism. The two values for the refractive index are, for the ordinary ray

$$\mu_o = 1.6585.$$

for the extraordinary ray

$$\mu_e = 1.4865.$$

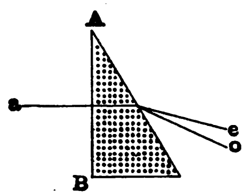


FIG. 149.

Should the angle of incidence change in this case, the direction

of the ray through the prism would be changed, but the light would at all times traverse the crystal at right angles to the axis, and the values of the indices for the two rays would remain constant.

(*c*) *The optic axis is normal to the refracting edge of prism, and parallel to first face of prism and to plane of incidence*, (Fig. 150). In this case so long as the incident light is normal to the first face of the prism, the same result will be obtained as in case *b*. Should the angle of incidence vary however, the direction of the light through the prism is no longer normal to the optic axis, and the index for the ordinary ray alone remains constant, 1.6585, while the extraordinary index varies from 1.4865 and 1.6585.

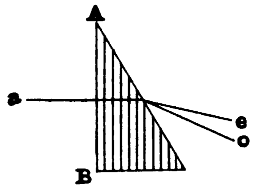


FIG. 150.

From this it is apparent that only in the second case would it is possible to determine the two indices by the method of minimum deviation, since in this case only does the light remain normal to the optic axis. In both the other cases the value of μ_e would vary between 1.4865 and 1.6585.

The results may be stated thus: A ray of light penetrating a crystal of Iceland spar is in general split into two rays. One, the ordinary ray, obeys the law of refraction and has always the same refractive index $\mu_o = 1.6585$. For the extraordinary ray, the value of the refractive index varies between 1.6585 and 1.4865. It assumes the maximum value whenever the ray follows the optic axis and the minimum value when it passes through the crystal at right angles to the optic axis.

218. Wave Surfaces in Uniaxial Crystals. Huygens explained the phenomena of double refraction in uniaxial crystals by an extension of the method adopted by him in the treatment of ordinary refraction. Since he had shown that the wave surface in an isotropic medium was a sphere, and since one of the rays in Iceland spar obeyed the laws of refraction in isotropic media, he assumed that for this ray the wave surface was a sphere. For the extraordinary ray he assumed the wave surface to be an ellipsoid of revolution about the optic axis, with its

center at the point of incidence, and having for one of its axes the diameter of the sphere. Between this axis and the second axis of the ellipsoid he assumed the same ratio to exist as existed between the velocities of the ordinary and extraordinary rays in the crystal. By means of these two surfaces the refracted waves may be found as in Article 193.

Thus let pp_2 (Fig. 151), represent a plane wave front incident upon AB, the surface of a uniaxial crystal whose optic axis pC, is assumed to lie in the plane of the paper. Then in time t, needed for the wave to travel the distance p_2p_2' *in air*, the spherical wave about p as center has reached Co, and the spheroidal wave has spread to Ce. Then by the principle of Huygens, tangent planes $p_2'o'o$ and $p_2'e'e$ from p'_2 upon the sphere and spheroid respectively, mark the wave fronts of the ordinary and extraordinary waves in the crystal.

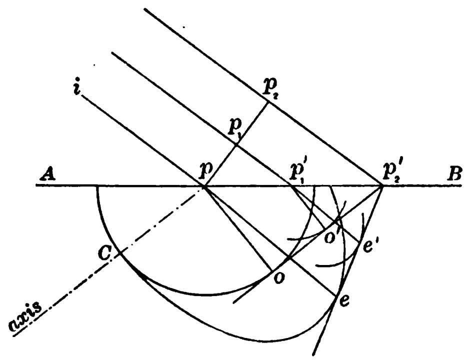


FIG. 151.

If the point of tangency to the sphere be o, then po is the ordinary ray, normal to the wave front. It lies in the plane of incidence and thus obeys both laws of refraction. If the tangent plane from p'_2 touch the spheroid at e, then pe is the extraordinary ray, which in general is not normal to the wave front and does not lie in the plane of incidence unless the optic axis is either *in* the plane of incidence or *normal* to it. In the figure the axis is assumed to lie in the plane of incidence and hence the extraordinary ray also lies in that plane and so obeys one of the laws of refraction.

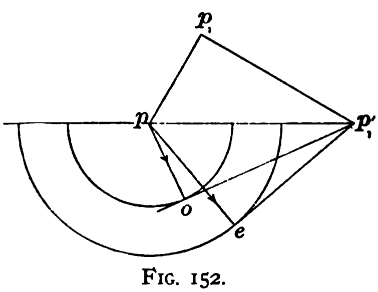


FIG. 152.

In one special case, however, when the optic axis is normal to the plane of incidence (Fig. 152), the extraordinary ray obeys both laws of refraction; that is, it is normal to the wave-front and lies in the plane of incidence, and consequently its velocity in the crystal bears a constant ratio to the velocity in air, for all angles of incidence. This ratio is termed the extraordinary index of refraction μ_e.

Thus (Fig. 152), the sections of the sphere and spheroid by the plane of incidence are both circles and if the velocity in air be taken as unity, and the velocities of the ordinary and extraordinary rays, as b and a respectively then

$$\mu_o = \frac{p_1p_1'}{po} = \frac{1}{b} = \mu_o, \text{ ordinary index}$$

$$\mu_e = \frac{p_1p_1'}{pe} = \frac{1}{a} = \mu_e, \text{ extraordinary index}$$

This case corresponds to (b) of Article 217, in which it was shown that the values of both refractive indices were constant for all angles of incidence.

Crystals in which the extraordinary index, μ_e, is greater than the ordinary index μ_o, are called *positive* crystals; those in which μ_e is less than μ_o are called *negative* crystals. In positive crystals such as quartz, ice and potassium sulphate, the ellipsoid lies within the sphere; in negative crystals, as Iceland spar, tourmaline, beryl, and sodic nitrate, the ellipsoid lies without the sphere.

CHAPTER XXVI.

EXPERIMENTAL DEMONSTRATIONS

219. The Nicol Prism. We have seen that a beam of plane polarized light may be produced in any one of a number of ways, as by reflection from a polarizing mirror at an angle of 57°, by a pile of plates, by a crystal of tourmaline, or by double refraction through a crystal of Iceland spar. To all these methods there are more or less serious objections. In the polarizing mirror it is difficult to secure an intense beam of polarized light, since only about 1/7 of the incident beam is reflected. In the pile of plates there is trouble from absorption by the plates, and diffusion of light from dust particles on the surfaces of the plates, thus causing stray light in the field. Tourmaline in plates of more than 2 mm. thickness absorbs the ordinary ray completely, but has the disadvantage that the extraordinary ray which is transmitted, is colored either green or red by the crystal, and also that tourmaline is not very transparent in plates of the required thickness.

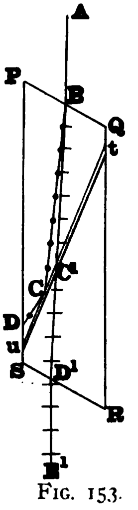


FIG. 153.

The most effective means of securing a strong beam of plane polarized light is by means of the Nicol prism. A clear crystal of Iceland spar is split out so that it is fully three times as long as it is broad. The end surfaces which in nature make an angle of 72° with the edges of the side, are so cut as to make the angle *SPQ* (Fig. 153), 68°. Let the section *PQRS* represent a principal section of the rhomb. The prism is then sawed in two along a plane normal to the new end surfaces and to the plane of the principal section *PR*. The two new faces are then polished and cemented together with Canada balsam, which has a refractive index smaller than that of the ordinary ray in the spar, but larger than that for the extraordinary ray. If now a ray *AB*, enter the rhomb parallel to its length the two rays are separated.

as usual in the spar, and the ordinary ray meets the Canada balsam at an angle slightly greater than the critical angle and is totally reflected through the side of the crystal and absorbed by its covering, which is painted dead black. The extraordinary ray passes into the Canada balsam as from a rarer to a denser medium and meets the second surface of the spar at an angle less than the critical angle from balsam to spar, and so is transmitted undiminished through the rhomb. As shown in the figure the vibrations in the extraordinary ray lie *in the principal section* through the shorter diagonal of the end surfaces of the rhomb. The Nicol thus transmits only those vibrations which are in the plane of its principal section, and quenches all vibrations at right angles to this plane.

220. Two Nicols. It is clear that if light emerging from one Nicol (polarizer), be passed through a second Nicol (analyzer), it will be transmitted if the principal sections of polarizer and analyzer be parallel, and will be totally extinguished if their planes be at right angles to each other. In any other position of the two Nicols there will be a portion transmitted and a portion absorbed. This is readily seen from Fig. 154.

Thus let a ray of plane polarized light, incident at O normally to the plane of the paper, have its vibrations parallel to OP, and let Op represent the amplitude of its vibration. Now if OA represent the principal plane of the analyzer, it is clear that the component of vibration parallel to this plane is Oa, the transmitted portion; while the normal component Ob is absorbed by the analyzer.

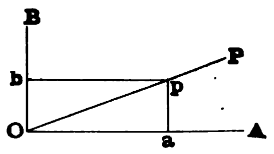


FIG. 154.

If $\alpha = POA$ be the angle between the planes of the two Nicols, then $Oa = Op \cos \alpha$ and the intensity of the transmitted component is proportional to $\cos^2 \alpha$.

If a plate G (Fig. 155), of any *isotropic* substance be placed in a beam of plane polarized light between crossed Nicols, the field remains dark for any position of the plate rotating about the beam ST as an axis. The reason for this is found at once in the fact that an isotropic substance affects in no wise the directions of vibrations of light transmitted by it; it is not doubly

refracting and hence the plane polarized light from the polarizer is transmitted through it unchanged and is extinguished by the analyzer.

221. Doubly Refracting Substance in Parallel Plane Polarized Light. If a thin plate of a doubly refracting crystal be interposed in a parallel beam of monochromatic light between crossed Nicols, there will be two positions of the plate in which the field will remain dark. These positions of the plate are those in which the two rectanglar directions of vibrations in the crystal coincide with the planes of vibration in the two Nicols. In these positions, the light from the polarizer is transmitted unchanged by the crystal and quenched by the analyzer. In any other position of the plate as it is rotated about the beam as an axis, the field will light up. This is because the plane polarized vibrations



FIG. 155.

from the polarizer are resolved by the plate into two component vibrations at right angles to each other, each of which will furnish a component parallel to the principal plane of the analyzer and so pass through, lighting up the field, while the other two components being at right angles to the plane of the analyzer, are quenched. If now white light be used instead of monochromatic light, the field will light up as before, on rotation of the crystal plate, but the light emerging from the analyzer will be colored and the color will depend upon the thickness of the crystal plate. If, on the other hand, the plate be fixed and the analyzer be rotated, the color fades until white is reached at the position of coincidence of the planes of the Nicols, after passing this position the color changes to the complementary hue and grows to a maximum at 45° from the position for white. This production of color from polarized light is due to interference. From the experiments of Fresnel and Arago, it was shown that two conditions were necessary for two beams of polarized light to interfere in the same way as in the case of ordinary light. *First, that the two beams of light shall be polarized in the same plane; second, that they shall have a common origin.*

Now the two rays into which the light from the polarizer is split up by the crystal, traverse the crystal plate with different velocities, and hence when the two components parallel to the plane of the analyzer are reunited, there will be a difference in phase, which for some color will amount to $\lambda/2$, the corresponding color will be absent, and the remaining light will be colored. The component from the extraordinary ray will be in advance of that from the ordinary ray, if the plate be from a *negative crystal*, it will be behind in phase if the plate be from a *positive crystal*.

222. Rings and Cross in Iceland Spar. If a thin parallel plate of Iceland spar or other uniaxial crystal be cut at right angles to the optic axis and interposed between crossed Nicols in a *convergent* beam of plane polarized light, there results a series of brilliantly colored concentric rings about a dark center and traversed by a dark cross as shown in Fig. 156. The axis of the convergent beam should strike the plate normally, in which case it is transmitted along the axis of the crystal, suffers no double refraction, and is quenched by the analyzer, thus forming the dark center. Since the plate is cut normally to the optic axis, all planes normal to the surface of the plate and passing through the axis are *principal sections*. The vibrations of the ordinary ray are *normal* to these planes or *tangential* to the system of circles, while the vibrations of the extraordinary ray are *in* these planes and hence *radial* to the circles. Now all rays of the convergent pencil, except the central one will pierce the plate at an angle to the optic axis, and hence would ordinarily undergo double refraction, but in two of these planes, representing the planes of vibration of the polarizer and analyzer, one of the components is wanting. Thus if the polarizer transmit only vertical vibrations, then in the vertical diameter the ordinary vibration is zero, and in the horizontal diameter the extraordinary vibration is wanting, thus leaving only vertical vibrations in these two diameters which are quenched by the analyzer. In all other directions the rays traversing the plate furnish both extraordinary and ordinary rays

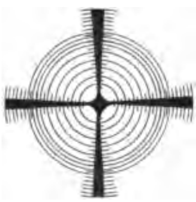


FIG. 156.

which after resolution by the analyzer produce components vibrating in the same plane but differing in phase, owing to different speeds of transmission through the plate. Now the locus of all points in the plate producing any definite difference of phase, due to the passing through a given thickness of the plate, will be a circle. Hence after the extinction of any wave length λ, there is left a circle of residual color through these points. Again since a smaller thickness is needed to produce a difference of phase of $\lambda/2$ for violet than for red, it follows that the violet is extinguished *first* and the *red* last in each successive ring of color. The rings of residual color are therefore red on their inner and violet on their outer edges. Also, since the thickness of the plate traversed increases with the obliquity of the rays, the successive rings grow narrower toward the edges and by overlapping soon produce uniform illumination. If the analyzer be rotated through 90°, the black cross is replaced by a white one, and the rings are seen projected upon a white field instead of upon a dark background. The effect of superposing upon a uniaxial plate cut normally to the optic axis, a second similar plate is the same as increasing the thickness of the first plate, *if they be from crystals of the same sign*. Hence a second negative plate laid upon the plate of Iceland spar would cause the rings to *contract*. A plate from a positive crystal would cause them to *dilate*. In this way the sign of a crystal may be determined by comparison with another crystal of known sign.

223. Double Refraction in Isotropic Media under Stress. If any isotropic substance be subjected to an unequal stress, the substance becomes doubly refracting and shows characteristic reactions when examined between crossed Nicols. Thus, if a piece of well annealed glass, which shows no double refraction, be placed between the Nicols and subjected to slight pressure or tension, a characteristic colored pattern at once appears. If a narrow glass plate be placed between the crossed Nicols and then set in longitudinal vibration by stroking it with a moist cloth, the field will light up at each stroke of the cloth, thus showing the effect of the alternate compressions and dilations at the nodes of the sounding glass plate. Kundt showed that the glass plate behaved as a positive crystal (quartz) when dilating and as a negative (Iceland spar) crystal when contracting.

Mach has shown that viscous substances, like Canada balsam, warm rosin, hot glass, etc., may be made doubly refracting for a few moments by pressure or distortion. The effect passes away as soon as the molecules readjust themselves. Kerr has also shown that fluid as well as solid dielectrics become doubly refracting when subjected to electrical stresses. When isotropic water freezes into ice the direction of the optic axis is coincident with the direction of the *stress* due to gravity.

224. Elliptic Polarization. We have seen that if a beam of plane polarized light fall upon a thin plate of doubly refracting crystal, the original vibration is resolved into two vibrations at right angles to each other. These two sets of vibrations travel through the plate with different velocities and consequently on emerging from the plate one vibration is behind the other in phase. Now *within the region in which the two rays of light overlap*, the ether particles are simultaneously subjected to these two simple harmonic vibrations at right angles to each other which have the same period, but which, in general, have different amplitudes and some definite difference in phase dependent upon the thickness of the plate. The motion resulting from compounding two such vibrations of the same period, is an elliptic motion. Hence the light emerging from the plate is said to be elliptically polarized *in the region where the two beams of light overlap*. This means that the paths of the ether particles transmitting the luminous disturbance are ellipses. Polarized light which has been reflected from a metal surface is also in general elliptically polarized. In the special case in which the two rectangular components have the same amplitude and a difference of phase of one-quarter of a period, the resultant is circular motion. Circularly polarized light is produced by passing plane polarized light through a sheet of mica of such thickness that the retardation of one set of vibrations with respect to the other, is *a quarter wave length*. Such a plate is called *a quarter wave plate*. Light which has been circularly polarized will appear equally bright for all positions of the analyzer. Elliptically polarized light shows maximum brightness when the plane of the analyzer is parallel to one axis of the ellipse, and minimum brightness when it is parallel to the other.

225. Rotary Polarization. If a plate of quartz cut normally to the optic axis be interposed between a pair of crossed Nicols the darkened field is at once lighted up. If monochromatic light be used the field can again be darkened by rotating the analyzer through a certain angle α. This shows that the light emerging from the quartz plate is still plane polarized, since it can be extinguished by the analyzer, but that the *plane of polarization has been rotated by the quartz plate through the angle* α. Experimental measurements have shown that the rotation is proportional to the *thickness of the quartz plate,* and *increases as the wave length decreases,* being about three times as great for violet light as for red. If therefore, white light be used and the quartz plate be interposed between the crossed Nicols, the field will be colored, the tint depending upon the thickness of the plate; the color will remain no matter how the quartz plate be rotated about the beam as an axis. If the analyzer be rotated, the separate colors may be extinguished, one after the other as the angle of rotation for the various colors is reached, but the field remains constantly lighted with the residual color. Some varieties of quartz rotate the plane of polarization *clockwise* and are termed *right-handed,* others rotate in the opposite direction and are called *left-handed* quartz. Besides quartz, many other crystals as cinnibar, sodic chlorate, and the hyposulphates of lead, calcium and potassium possess this rotary power. Not only this, but even liquids and vapors have been found to possess rotary power, though in a much less degree. This seems the more remarkable, in view of the fact that *crystals when fused lose their rotary power.* Of the rotary active substances, perhaps the sugars are the most important, and the methods of testing and determining percentages of sugar in solution, forms one of the most important commercial applications of polarized light.

226. Magneto-Optical Rotation. In 1845, Farraday discovered that isotropic substances, especially substances having high refractive power such as dense glass, were capable of rotating the plane of polarization of light when they were placed in a strong magnetic field. The rotation is *clockwise, to a person looking along the lines of force, and in the direction of propagation of the light.* When the light is made to retrace its path by

reflection, the direction of rotation is reversed and so *the effect is increased proportionally to the number of times the light traverses the isotropic substance.* No effect is produced if the light pass at right angles to the lines of force. The radical difference between this and other rotary phenomena is, that in crystals or liquids the rotation produced by passing a beam through the substance is reduced to zero if the ray be made to retrace its path, while in the case of a magnetic field the rotational effect is doubled.

ON THE USE OF LOGARITHMIC TABLES

Logarithms are used for lessening the drudgery of extended mathematical computations. By means of the ordinary logarithms the operations of multiplication, division, involution and evolution are reduced to those of addition, subtraction, multiplication and division.

(*a*) In the use of common logarithms, it is constantly to be borne in mind that every number is to be regarded as some power of the base number, 10, and that *the logarithm is the exponent indicating that power.*

Thus $10 = 10^1$ or $\log 10 = +1$, also $\log 0.1 = -1$,
$100 = 10^2$ $\log 100 = +2$, $\log 0.01 = -2$,
$1000 = 10^3$ $\log 1000 = +3$, $\log 0.001 = -3$, &c.

(*b*) Again since any number affected with the exponent zero is equal to unity, it follows that $10^0 = 1$, or $\log 1 = 0$. From this it appears that the logarithm of any number between one and ten is greater than zero, and less than one; that is, it is *some positive fraction.*

Similarly the logarithm for any number between 10 and 100 is +1 plus some fraction, for any number between 100 and 1000 the corresponding logarthm is +2 plus some fraction, and so on. In the same way it is evident that for any number between 0.01 and 0.1 the logarithm is —2 plus some fraction, for any number between 0.1 and 1 the logarithm is — 1 plus some fraction, etc.

(*c*) The integral part of the logarithm is called *the characteristic,* and is always one less than the number of figures standing to the left of the decimal point. The characteristic of a fractional number expressed as a decimal is negative and *one more* than the number of ciphers preceding the first significant figure. The fractional part of the logarithm is usually expressed as a decimal, and taken so as to be positive. It is called *the mantissa,* and is always the same for the same series of figures, regardless of the position of the decimal point. Thus the logarithms of the following numbers differ *only in the characteristic,* the mantissa being the same throughout.

Number.	Logarithm.	
37684	4.5762	
3768.4	3.5762	
376.84	2.5762	
37.684	1.5762	
3.7684	0.5762	
0.37684	—1.5762	or more usually written 9.5762
0.037684	—2.5762	or more usually written 8.5762 etc.

From the foergoing examples it will be seen that negative characteristics are avoided by adding 10 to the logarithm and afterward dropping ten from the logarithm of the result. No confusion can arise from this if the student will but stop to think that adding ten to the characteristic of the logarithm of a number, is in reality multiplying the number in question by 10^{10}, and the uncorrected result would therefore be 10,000,000,000 times too large, a fact that would certainly attract attention.

To find the logarithm of a number. First determine *the characteristic* from the rule given under (*c*) and then for *the mantissa* refer to the tables. If the number contain not more than two figures it will be found in the column headed in N, and its mantissa will be found directly opposite in the column headed 0. If the number contain three figures, the

first two will be found as before, and the third in the horizontal space at the top of the table; the mantissa of the number will be found on the horizontal line containing the first two figures, and in the vertical column headed by the last figure. Thus log 2 = 0.3010, log 7 = 0.8451, log 35 = 1.5441, log 353 = 2.5478, etc. If the number contain more than three figures, find the mantissae corresponding to the round numbers below and above the number, expressed in three significant figures; the difference between these two mantissae is called *the tabular difference*. The mantissa is now to be increased by a certain part of this tabular difference. Thus to find the logarithm of 35367 we proceed as follows:

log 35400 = 4.5490

log 35300 = 4.5478, their difference is 0.0012. Now in order to obtain the logarithm of 35367 we must add to the logarithm of 35300, 0.67 of the tabular difference, or $0.67 \times 0.0012 = 0.0008$ to the nearest integer. Hence log 35367 = 4.5478 + 0.0008 = 4.5486.

To find the number from the logarithm the process is simply reversed. Thus to find the number corresponding to the logarithm 4.5486 we find the mantissa next lower to correspond to 35300, the difference between this and the given logarithm is 0.0008 and the tabular difference is 0.0012. The last two figures are gotten by dividing 8 by 12, giving 67 as before.

From the fundamental properties of logarithms we have the following obvious relations:

$$\log ab = \log a + \log b$$
$$\log ab = \log a - \log b$$
$$\log a^{n} = n \log a$$
$$\log \sqrt[n]{a} = \frac{1}{n} \log a.$$

Hints on Computation. It is usually easier to add a series of numbers correctly, than to perform an equal number of subtractions. For this reason the *arithmetical complement* or *co-logarithm* of any number occurring as a divisor, is used and addition employed instead of subtracting logarithms. The co-logarithm of any number is obtained by subtracting the logarithm of the number from ten, thus avoiding negative characteristics. It is also to be observed that the co-logarithm of any number is the logarithm of the reciprocal of the number. Thus log 2 = 0.3010, and co-log 2 = 9.6990. All logarithms of numbers should be clearly labelled, so that once found, the same logarithm need not be looked out again.

All work should be tabulated and arranged in orderly fashion, before any values are taken from the table.

Example. Let it be required to compute f from the formula

$$f = \frac{4\pi^2 r}{T^2}$$

where $r = 3.84 \times 10^{10}$ cm. and $T = 27$ da. 8 hr. $T = 2.3616 \times 10^{6}$ sec.

$\log \pi =$	0.4971		
$\log \pi^2 =$	0.9943		
$\log 4 =$	0.6021	$\log T =$	6.3732
$\log r =$	10.5843	$\log T^2 =$	12.7464
co-log $T^2 =$	7.2536	colog $T^2 =$	7.2536
$\log f =$	9.4343		
$f =$	0.271 cm./sec.^{2}		

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774

N	0	1	2	3	4	5	6	7	8	9
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	00000	0043	0087	0130	0173	0217	0260	0303	0346	0389
101	00432	0475	0518	0561	0604	0647	0689	0732	0775	0817
102	00860	0903	0945	0988	1030	1072	1115	1157	1199	1242
103	01284	1326	1368	1410	1452	1494	1536	1578	1620	1662
104	01703	1745	1787	1828	1870	1912	1953	1995	2036	2078
105	02119	2160	2202	2243	2284	2325	2366	2407	2449	2490
106	02531	2572	2612	2653	2694	2735	2776	2816	2857	2898
107	02938	2979	3019	3060	3100	3141	3181	3222	3262	3302
108	03342	3383	3423	3463	3503	3543	3583	3623	3663	3703
109	03743	3782	3822	3862	3902	3941	3981	4021	4060	4100
110	04139	4179	4218	4258	4297	4336	4376	4415	4454	4493

INDEX

Numbers refer to pages

Publications of George Wahr, Ann Arbor

DZIOBEK.—*Mathematical Theories of Planetary Motions.* By Dr. Otto Dziobek, Privatdocent in the Royal Technical High School of Berlin, Charlottenburg. Translated by Mark W. Harrington, formerly Chief of the United States Weather Bureau, and Professor of Astronomy and Director of the Observatory at the University of Michigan, President of the University of Washington, and Wm. J. Hussey, Assistant Professor of Astronomy in the Leland Stanford, Jr. University. 8vo. 294 pages. $3.50.

GUTHE.—*Laboratory Exercises with Primary and Storage Cells.* By Karl E. Guthe, Assistant Professor of Physics, University of Michigan. 58 pages. 17 illustrations. Cloth, 45 cents.

KRAUS.—*Essentials of Crystallography.* By E. H. Kraus, Junior Professor of Mineralogy in the University of Michigan. 300 pages. 427 illustrations. Large Octavo. $1.60.

MONTGOMERY-SMITH.—*Laboratory Manual of Elementary Chemistry.* By Jabez Montgomery, Ph.D., Professor of Natural Science, Ann Arbor High School, and Roy B. Smith, M.S., Colgate University. 12mo. 400 pages. Cloth, $1.25.

NETTO.—*The Theory of Substitutions and its Application to Algebra.* By Dr. Eugene Netto, Professor of Mathematics in the University of Giessen. Revised by the author and translated with his permission, by F. N. Cole, Ph.D., formerly Assistant Professor of Mathematics in the University of Michigan, Professor of Mathematics, Columbia University. 8vo. 301 pages. Cloth, $3.00.

REED-GUTHE.—*A Manual of Physical Measurements.* By John O. Reed, Junior Professor of Physics, University of Michigan, and Karl E. Guthe, Professor of Physics in the State University of Iowa. 185 pages. 89 illustrations. Octavo. $1.60.

REED.—*College Physics. For Students in Academies and Colleges. Physics I. Mechanics—Sound—Light.* By John O. Reed, Junior Professor of Physics, University of Michigan. Octavo. 300 pages. Cloth, $1.50.

WRENTMORE.—*Plain Alphabets for Office and School.* Selected by C. G. Wrentmore, B.S., C.E., Instructor in Descriptive Geometry and Drawing, University of Michigan. Oblong. 19 plates. Half leather, 75 cents.

ENGINEERING LABORATORY NOTE BOOK.—*A Note Book for the Engineering Laboratory, University of Michigan.* Full sheep binding. Size 5½ x 8. Contains 200 pages. (With general directions. Cross section ruled). Price, 75 cents.

FIELD ENGINEERING NOTE BOOK, SURVEYING.—200 pages. Cross section ruled. Full sheep binding. 50 cents.